Human Capital, Stochastic Growth, and Welfare

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Dr. Kuwahara taught me, when a Master student, graduate-level macroeconomics based on Bénassy (2011) and Aghion and Howitt (2009), and I could learn a lot about growth theory. At last, the model of this thesis is somehow not covered in neither of these textbooks; I told him that I would be interested in economic development, and he taught me the Uzawa-Lucas model – the model I am using in this thesis throughout. Therefore, I doubt whether I could complete this thesis without his teaching. Furthermore, when he started calculation in front of me, I could "feel" the role of theory in economics.

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Last but not least, I would like to thank and dedicate this thesis to my family. Thank you very much for your constant support, encouragement, and unending love. My father, Junichi, my mother, Megumi, and my little brother, Asato, have always helped me move on – without you, this thesis would not have been possible.

To my family...

Abstract

Will higher uncertainty speed up or slow down growth? Empirical studies suggest four links between growth and uncertainty; their relationship is (i) negative (ii) positive (iii) U-shaped and (iv) inverted U-shaped. To account for these conflicting facts, I analytically analyze the two-sector, endogenous growth model featuring human capital accumulation and various types of uncertainty. I show that the model can replicate all four patterns, hence shedding analytical light on divergent empirical evidence on the growth-uncertainty nexus.

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Chapter 1

Introduction

In 2017, an estimated *6.3 million* children and young adolescents died mostly from *preventable* causes.[1] A woman in sub-Saharan Africa has a one-in-*thirty* chance of dying while giving birth; in the developed world, the chance is one in *5,600*. There are at least 25 countries where the average person is expected to live *less than 55 years*. In India alone, more than *50 million* school-going children *cannot* read a very simple text.[2]

Unwilling to live with the injustice we see in the world, economic growth has been one of the most active fields of research in economics, in particular since the mid-1980s.[3] As economic policy makers constantly shape the course of growth and development (Jones, 2002, p.3), the goal of research on economic growth is to provide a general economic framework to help us understand the process of growth and development; a prerequisite to better policies to eliminate the injustice is a better understanding of economic growth.

Research on economic growth has made important advances over the past decades, including the several excellent ideas – for example, of Lucas (1988, 2002) and Romer (1990) – that have already earned Nobel Prizes. Even though "economic growth is necessary but not sufficient for poverty to fall," (Giugale, 2017, p.5) we require models with stronger theoretical foundations to gain a better un-

[1] United Nations Inter-agency Group for Child Mortality Estimation (2018, p.6). Children under age 5 accounted for 5.4 million of these deaths.

[2] These data are from Banergee and Duflo (2011, p.1). Weil (2013) and Jones (2016) provide an abundance of the facts of economic growth.

[3] See the Introduction of Barro and Sala-i-Martin (2004) and Turnovsky (2009, Ch.1) for a nice discussion of the history of research on economic growth.

derstanding of the process of economic growth. The wealth of growth models is available in Acemoglu (2009) and Aghion and Howitt (2009); but there is so much that we still need to know. And the purpose of this thesis is to analyze one of such unknowns: *stochastic* growth – the impact of uncertainty on growth.[1] Put differently, this thesis answers the question: "Will higher uncertainty speed up or slow down economic growth?"

We begin with brief review of empirical studies. We then examine a number of the relevant facts that describe the relationship between growth and uncertainty, keeping in mind that the latter cannot do justice to the former.

1.1 Literature

The seminal paper of Ramey and Ramey (1995) proposes the empirical framework to analyze the relationship between growth and its standard deviation (called *uncertainty* in what follows). They began with a critique on the standard dichotomy of macroeconomics: business cycle fluctuations has *no* effect on growth. In a sample of 92 countries for the period 1960-1985, as well as a sample of 24 OECD countries from 1950 to 1988, they found that countries with higher uncertainty had lower growth; growth and uncertainty are *negatively* linked. Therefore, a policy designed to decrease the business cycle fluctuations is consistent with the goal of a high long-run growth (Norrbin and Yigit, 2005, p.343).

The Ramey-Ramey framework has been commonly used and extended by the profession over the past two decades or so, resulting in the wealth of empirical evidence on this subject. The essence of this large literature on growth and uncertainty can be simply summarized as in Fig. 1.1 – their relationship is (i) negative (ii) positive (iii) U-shaped and (iv) inverted U-shaped; that is, there are *four* patterns. So, the agenda of empirical studies is to turn up unquestionable evidence to reach an empirical consensus for the design of policy. In light of Fig. 1.1, let

[1] Three remarks in advance: (i) In line with empirical studies reviewed below, I only analyze the effects of uncertainty on growth, *not* of growth on uncertainty (ii) Throughout, I don't draw a distinction between "uncertainty," "risk," "volatility," and "shock." Empirical studies often use the word "volatility." For ease of exposition, however, I often use the word "uncertainty." (iii) Though a framework used in *short-run* business cycle research such as real business cycle (RBC) models (Cooley, 1995) or New Keynesian models (Galí, 2015) is also a stochastic growth model, my focus is instead on the *long-run* growth.

us review some papers that fall into each category.

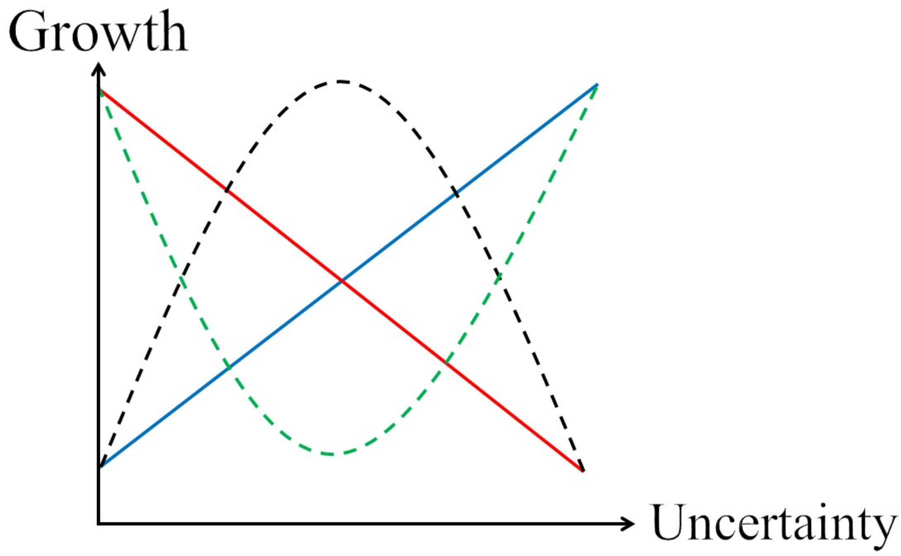

Figure 1.1 Growth and uncertainty: four links.

First, many studies have confirmed the *negative* link of Ramey and Ramey (1995). For example, Badinger (2010) proposes a new instrument to identify the effect of uncertainty on economic growth, and finds the negative relation in a sample of 128 countries between 1960 and 2003. Posch and Wälde (2011) also support a negative link by developing a stochastic "vintage capital" style growth model and examining 20 countries from 1970 to 2009 with taxes included as an important control variable. Berument et al. (2012), using an exponential generalized autoregressive conditional heteroskedasticity (EGARCH) model, confirm a negative link by analyzing the quarterly data for Turkey from 1987Q1 to 2007Q3 as well.[1]

Second, some studies have instead supported a *positive* relationship.[2] Caporale and McKiernan (1998), using an ARCH-M model, find a significant and positive link by analyzing the annual U.S. data from 1870 to 1993. Fountas and Karanasos (2006), covering the data for G3 (U.S., Japan, and Germany) over one and a half centuries, use a GARCH model and confirm a positive relation too. Moreover,

[1]Dawson and Stephenson (1997), however, show that the results of Ramey and Ramey (1995) disappear when the U.S. *state* level data are used for the period 1970-1988. Moreover, though Norrbin and Yigit (2005) support a negative link, they point out that it is more likely to hold among *non*-OECD countries.

[2]For this possibility, Bloom (2014) argues that, in theory, higher uncertainty may *stimulate* R&D; in the face of a more uncertain future, some firms may be more willing to innovate *now*, hence higher growth.

Imbs (2007) examines the *sectoral* data for manufacturing activities at the three-digit level in 47 counties, and argues that the relationship is negative between countries, but positive between sectors. Besides, Lee (2010) reports a positive link by using a dynamic panel GARCH model for G7 from 1965 to 2007.

Third, unlike studies cited above, two papers point out a *nonlinear* relationship. Examining the cross-country data for 114 countries between 1978 and 2002, García-Herrero and Vilarrubia (2007) demonstrate that there seems to exist the threshold at which the relation is reversed; the link between growth and uncertainty is *inverted U-shaped*. For example, as long as uncertainty is moderate, decreases in the business cycle fluctuations achieve a higher long-run growth. But when it exceeds its threshold level, a government policy that attempts to stabilize the business cycle of a country may damage its long-run growth potential. As such, they argue that the relationship between growth and uncertainty looks like the *Laffer curve*. Furthermore, recently, Alimi (2016) analyzes the growth-uncertainty nexus in a panel of 47 developing countries over the period 1980 to 2013, and supports the *U-shaped* (or the reversed Laffer curve) relation if uncertainty is less than 4%. As a result, the design of optimal growth policy is so complicated; we may be required to precisely estimate the threshold value of uncertainty to avoid policy mistakes.

Taking stock, as Fig. 1.1 displays, the empirical literature offers *four* possible links between growth and uncertainty. Next, we seek to understand more about them by looking at a set of scatter plots. As Norrbin and Yigit (2005) point out, the results of this literature seem to be sensitive to the choice of data − for example, countries and time periods. Even though scatter plots are not rigorous econometric output, they would be useful in grasping the true nature of this complex literature, to some extent.

1.2 Correlates

We start to examine an association between economic growth and uncertainty. Given the lack of an empirical consensus, we analyze three sets of groups in turn: BRICS, G7, and OECD. Unless stated otherwise, data are from *Penn World Table*

(PWT) version 9.0, a database with information on relative revels of income, output, input and productivity, covering 182 countries from 1950 to 2014.[1]

1.2.1 BRICS

BRICS consists of Brazil, Russia, India, China and South Africa. As there are only 5 countries, we conduct a time series analysis for BRICS; that is, we in turn look at the data of each country over time (in 5-year interval). Figs. 1.2 to 1.6 show a relationship between growth and uncertainty.[2] Each figure contains three elements: a straight regression line, a quadratic approximation curve, and a correlation coefficient. The reason for the first and third is obvious; the reason for the second is inspired by the *nonlinear* possibility of García-Herrero and Vilarrubia (2007) and Alimi (2016). Though we'll pay little attention to a quadratic curve, it may be meaningful when a curve is sufficiently convex (or concave). For convenience, correlation coefficients and their significance (based on p-value) are summarized in Table 1.1.

Table 1.1: Correlation Coefficients and Their Significance: BRICS

Country	**Correlation Coefficient**	**p-value**
Brazil	-0.16	0.63
Russia	-0.04	0.96
India	-0.08	0.79
China	-0.85	0.00
South Africa	-0.30	0.34

It is clear from Table 1.1 that we see *negative* relationships between growth and uncertainty, as Ramey and Ramey (1995) confirm. Their significance, however, varies across countries; the negative correlation is only significant (at 1%) in China. Though only suggestive, this observation illustrates how difficult to reach an empirical consensus is – whether you use the time series data for (say) China or Russia makes huge differences.

At the same time, Table 1.1 doesn't capture the potential nonlinearity. For example, Figs. 1.3 (Russia) and 1.6 (South Africa) imply an inverted U-shaped

[1] Available at `www.ggdc.net/pwt`; see Feenstra et al. (2015) for details.

[2] I basically use Stata 15 to produce figures. When I instead use Matlab R2019a, I will indicate that.

link; while Fig. 1.2 (Brazil) suggests a U-shaped association. To evaluate their true relevance, we need a rigours econometric test like García-Herrero and Vilarrubia (2007) and Alimi (2016). These scatter plots, however, may be telling us to consider a nonlinear possibility in a theoretical analysis below.

The bottom line of our BRICS analysis is that China exhibits a significant, negative relationship; Brazil a U-shaped; Russia and South Africa an inverted U-shaped.

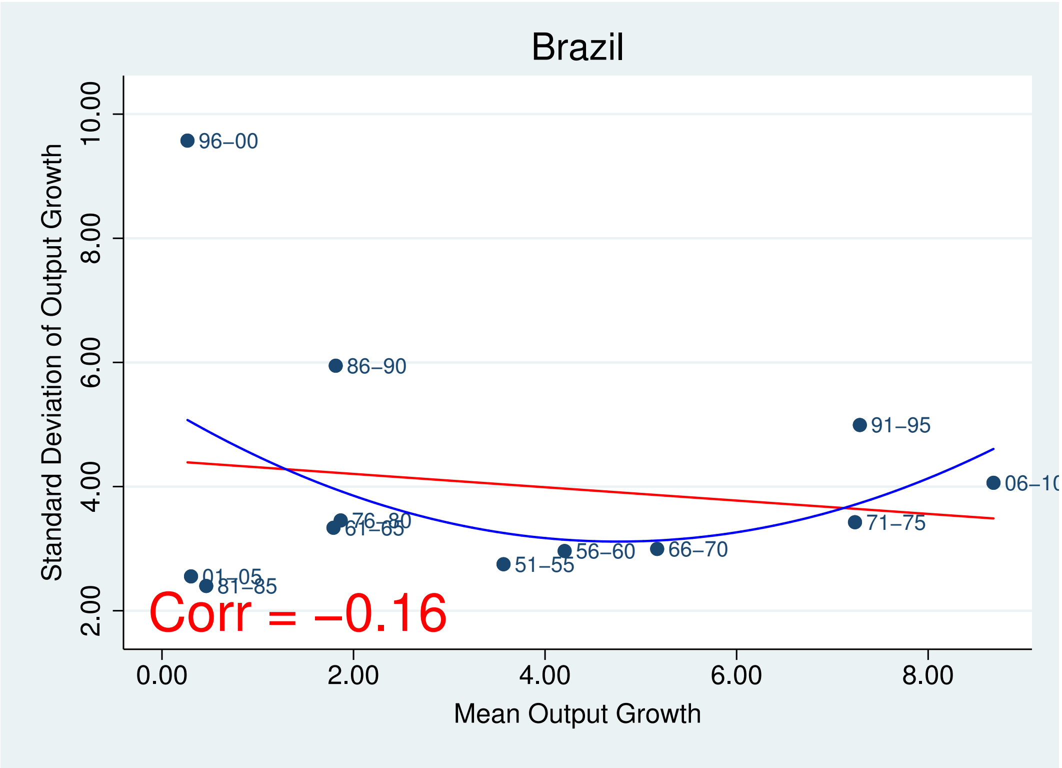

Figure 1.2 Growth and uncertainty: Brazil.

1.2.2 G7

Next, let us examine G7. It consists of Canada, France, Germany, Italy, Japan, the United Kingdom, and the United States. Just like a BRICS analysis, we undertake a time series analysis for each country over time, again in 5-year interval. The results are in Figs. 1.7 to 1.13, and the relevant statistics are summarized in Table 1.2.

According to Table 1.2, except for Italy, we gain see negative links. In the case of G7, only Canada exhibits a significant (at 1%), negative relation with a

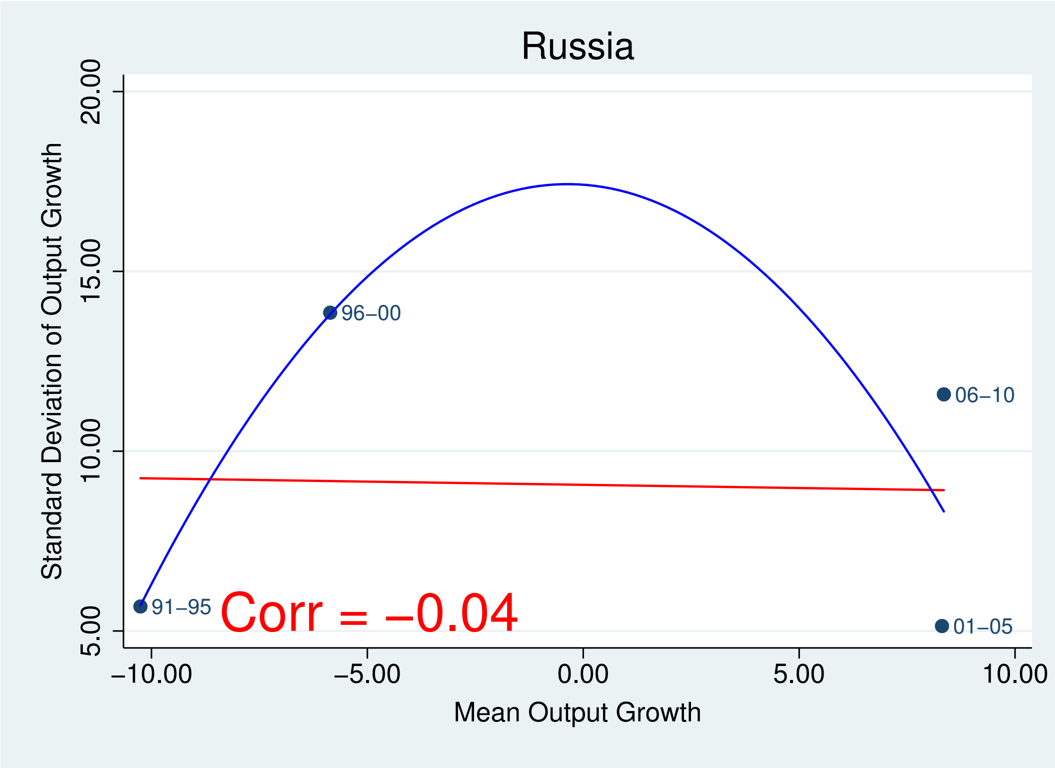

Figure 1.3 Growth and uncertainty: Russia.

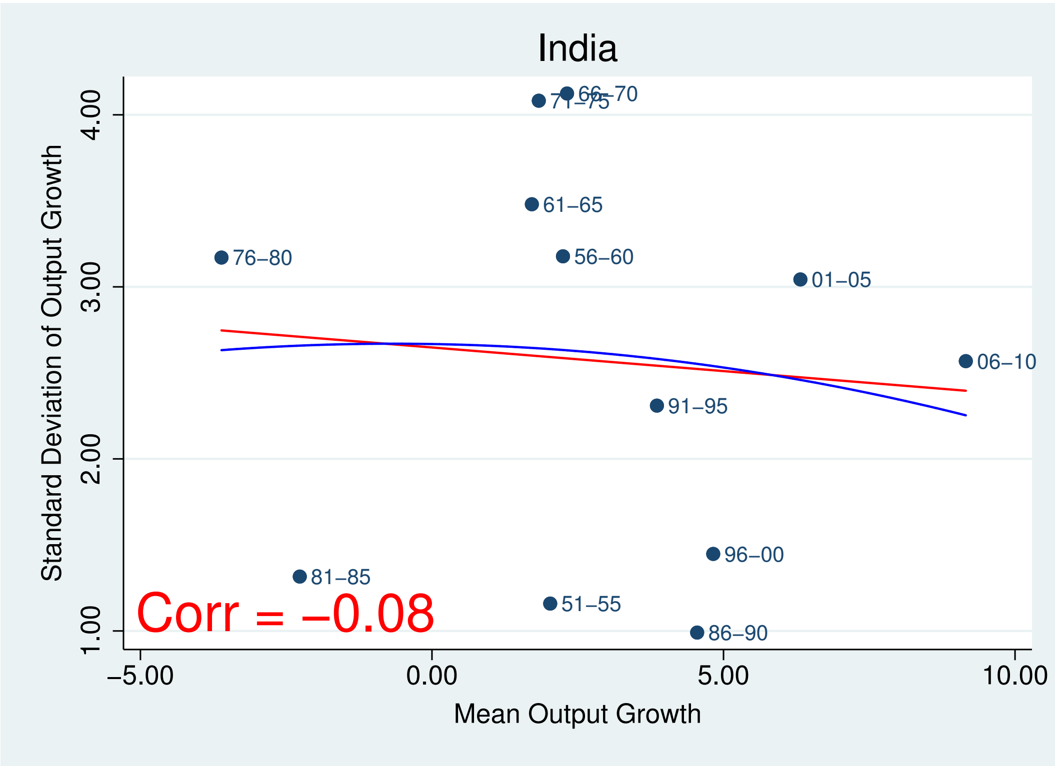

Figure 1.4 Growth and uncertainty: India.

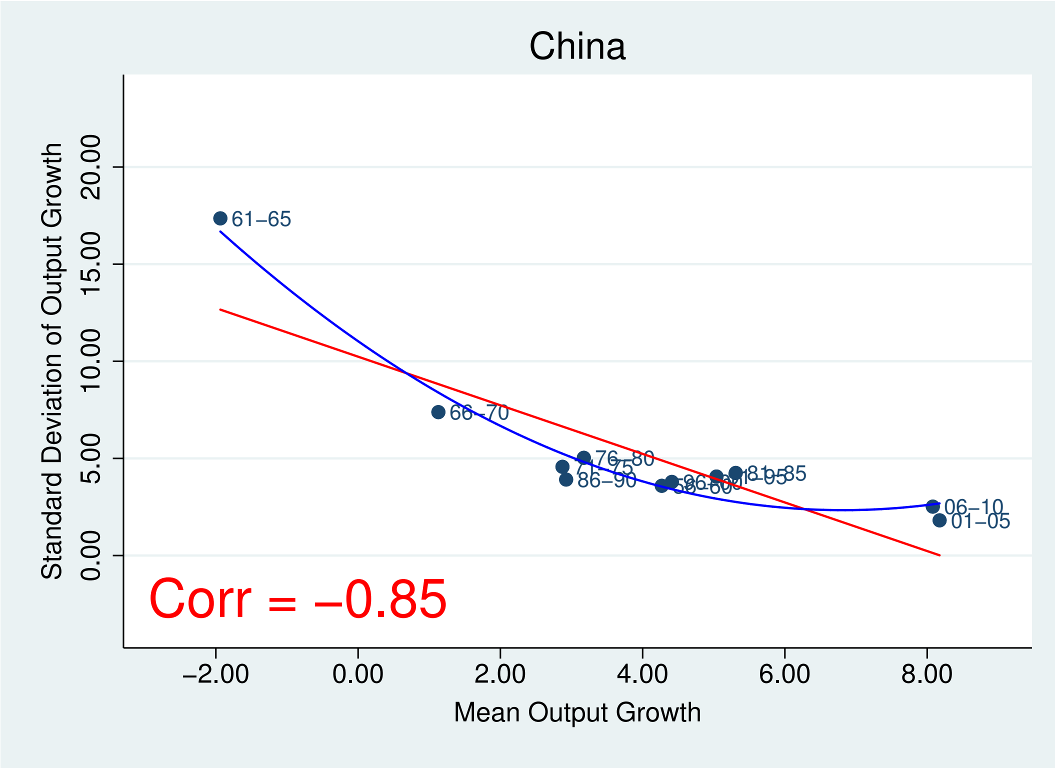

Figure 1.5 Growth and uncertainty: China.

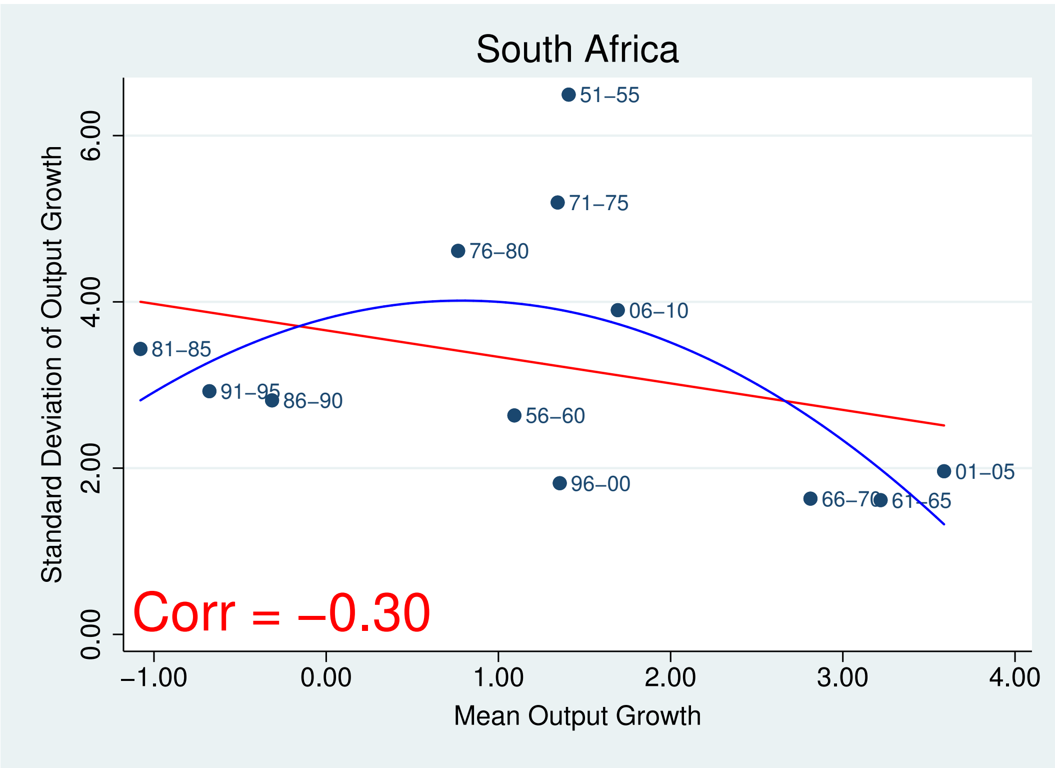

Figure 1.6 Growth and uncertainty: South Africa.

Table 1.2: Correlation Coefficients and Their Significance: G7

Country	Correlation Coefficient	p-value
Canada	−0.88	0.00
France	−0.12	0.70
Germany	−0.02	0.95
Italy	+0.01	0.99
Japan	−0.33	0.30
United Kingdom	−0.42	0.17
United States	−0.26	0.41

correlation coefficient of −0.88: a strong correlation. Thus, in Canada, higher growth tends to introduce higher uncertainty, or vice versa (or both). In terms of nonlinearity, only France (Fig. 1.8) exhibits an inverted U-shaped association between growth and uncertainty. Therefore, in France, it may better be captured by a quadratic approximation than a linear one.

The bottom line of our G7 analysis is that Canada exhibits a significant, clear negative relationship; and France an inverted U-shaped.

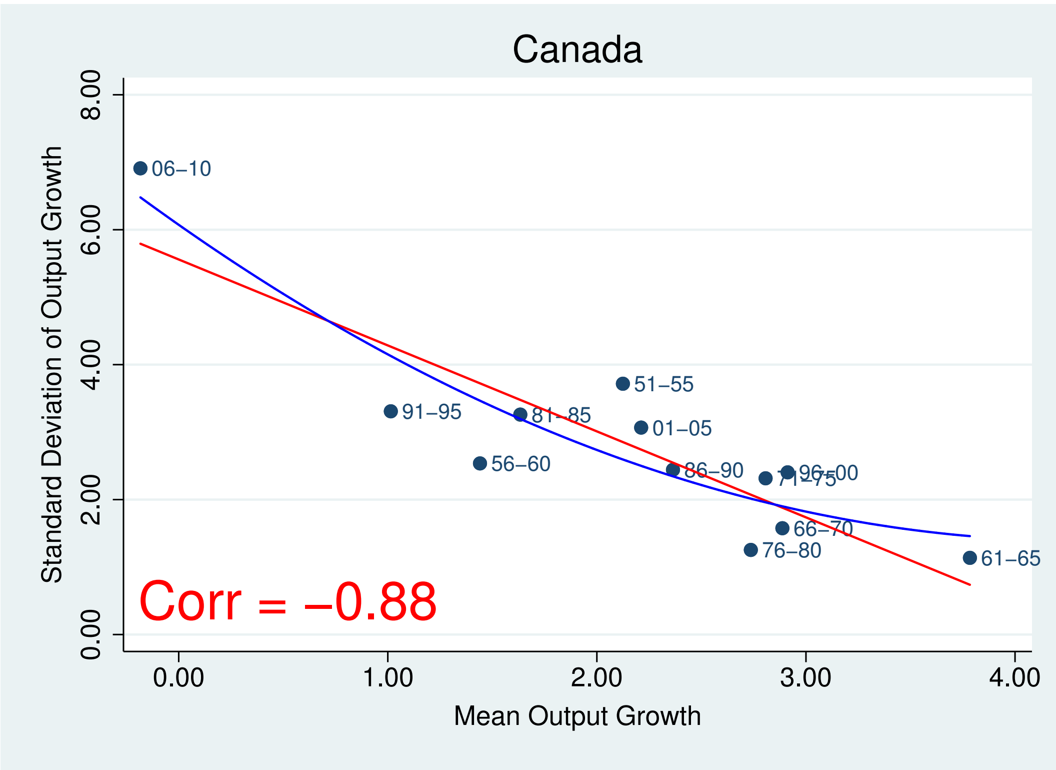

Figure 1.7 Growth and uncertainty: Canada.

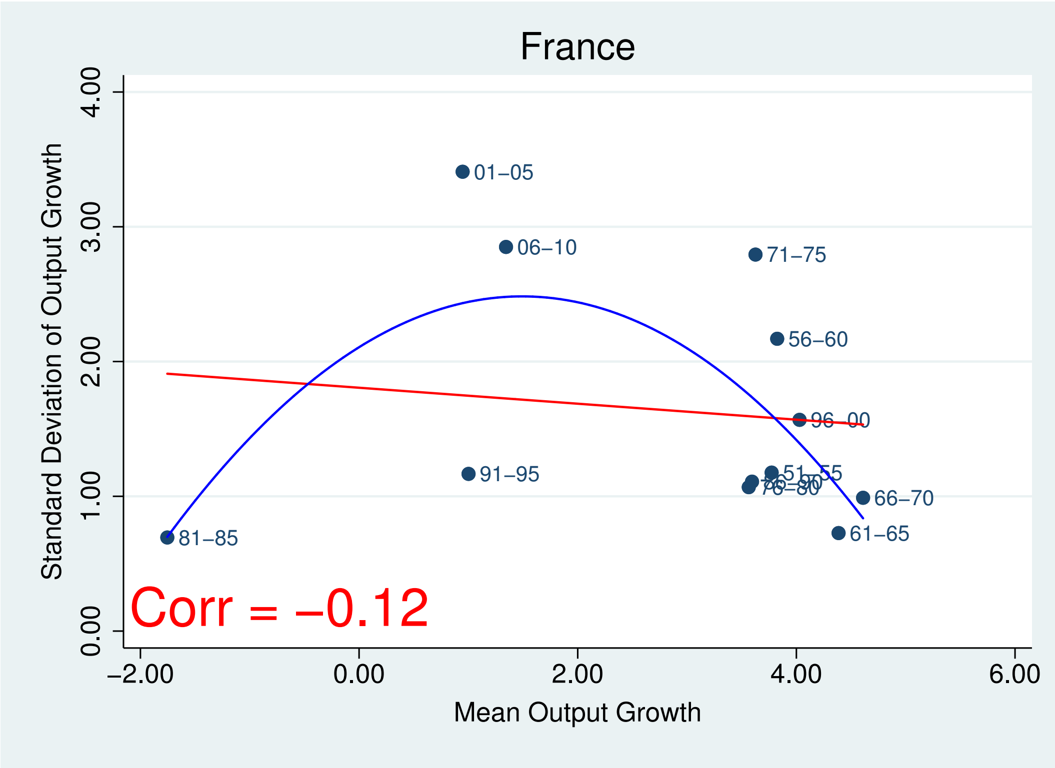

Figure 1.8 Growth and uncertainty: France.

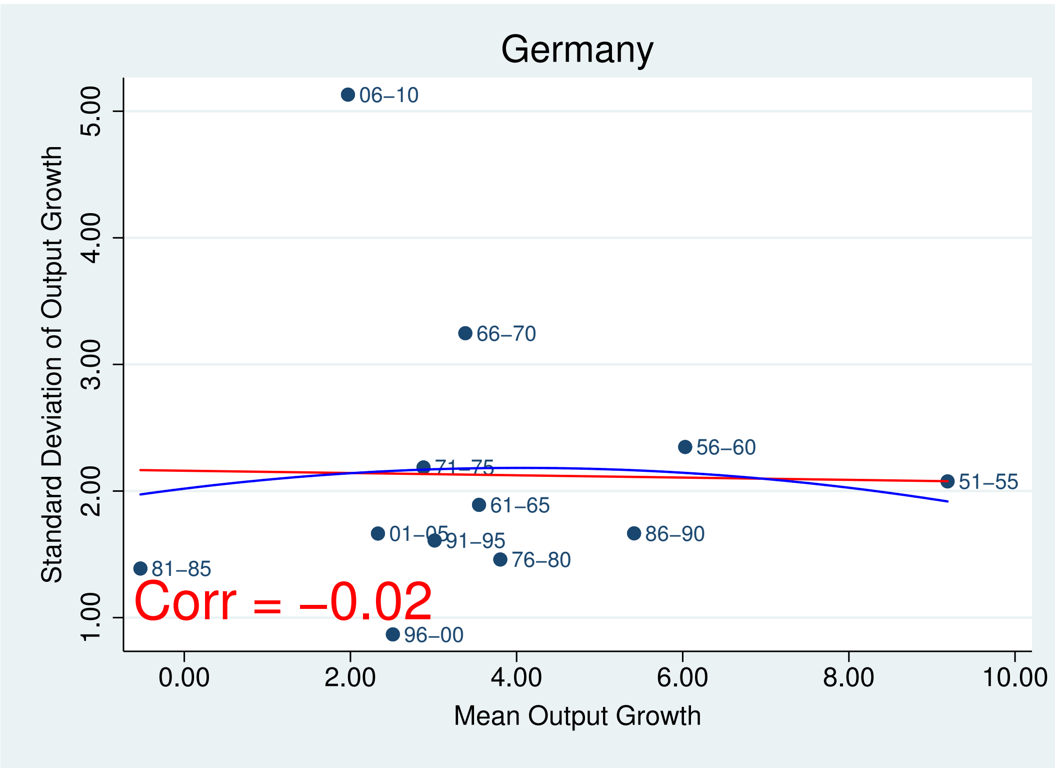

Figure 1.9 Growth and uncertainty: Germany.

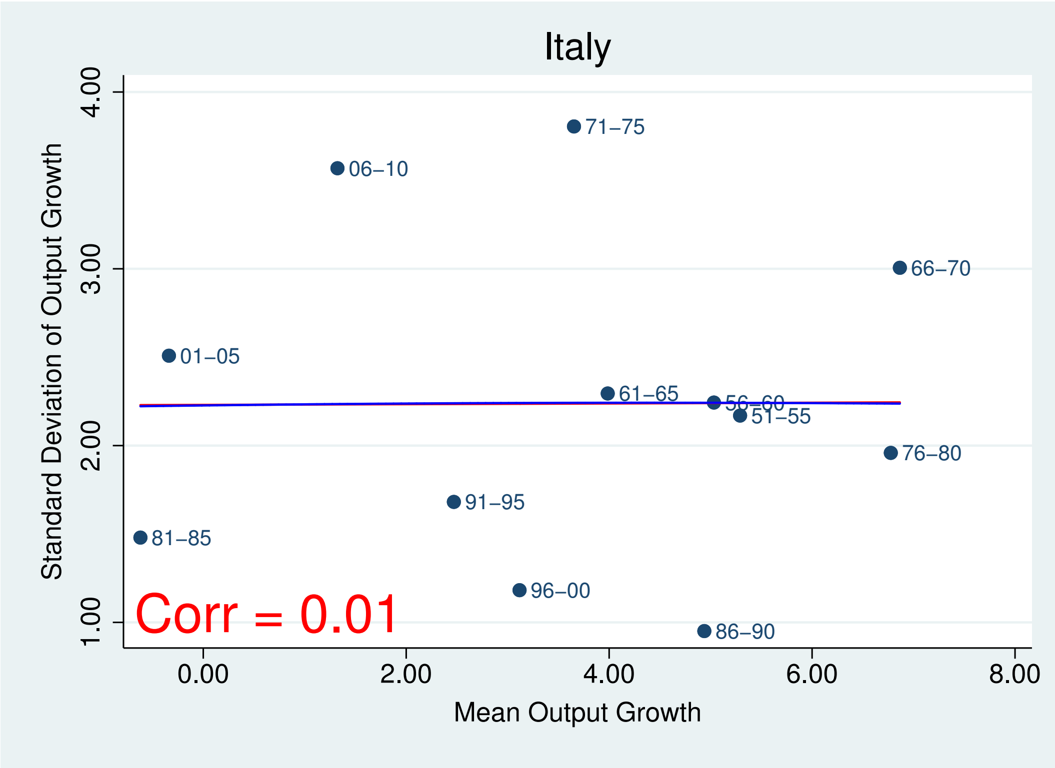

Figure 1.10 Growth and uncertainty: Italy.

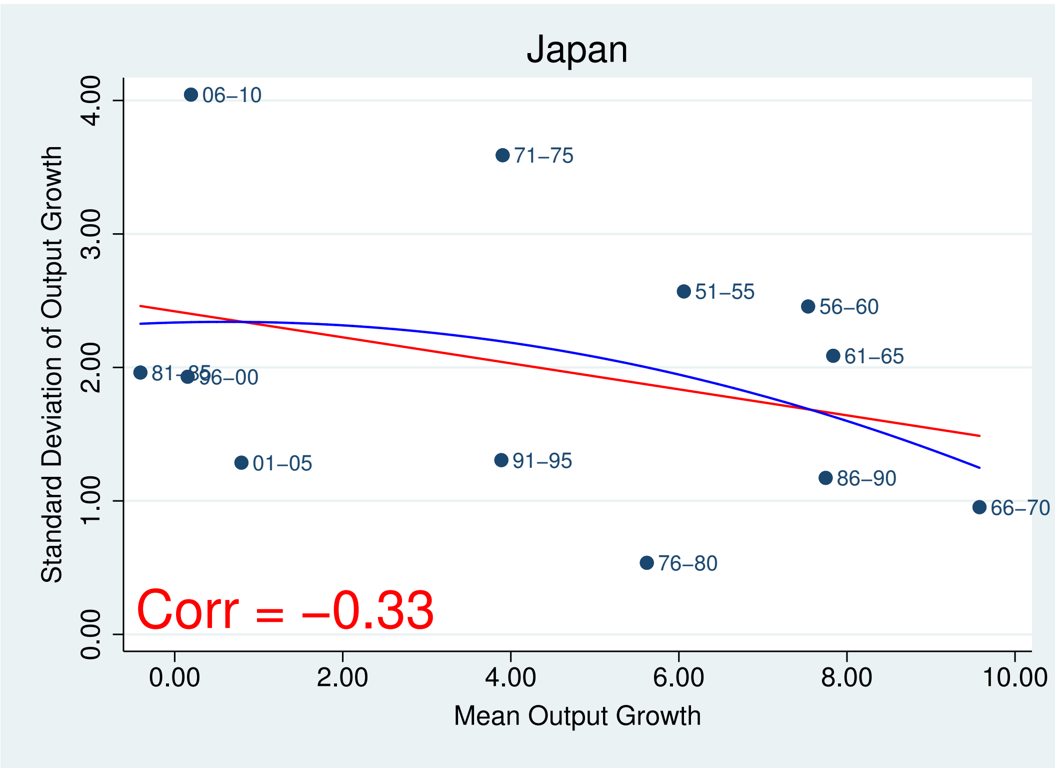

Figure 1.11 Growth and uncertainty: Japan.

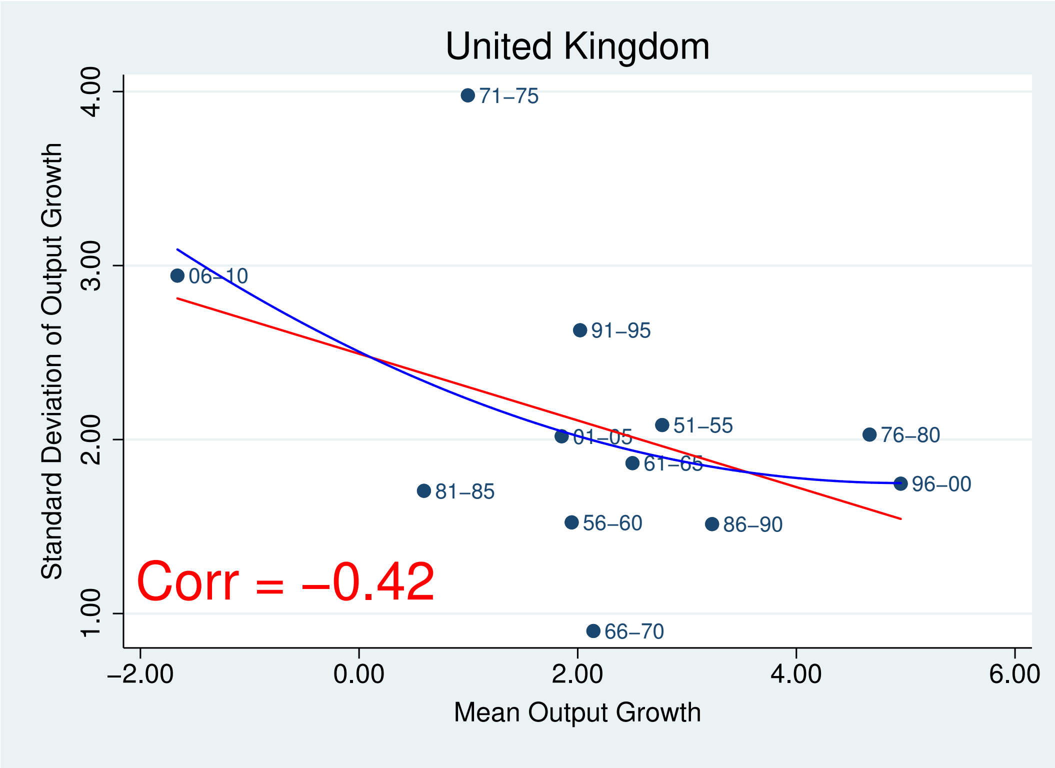

Figure 1.12 Growth and uncertainty: the United Kingdom.

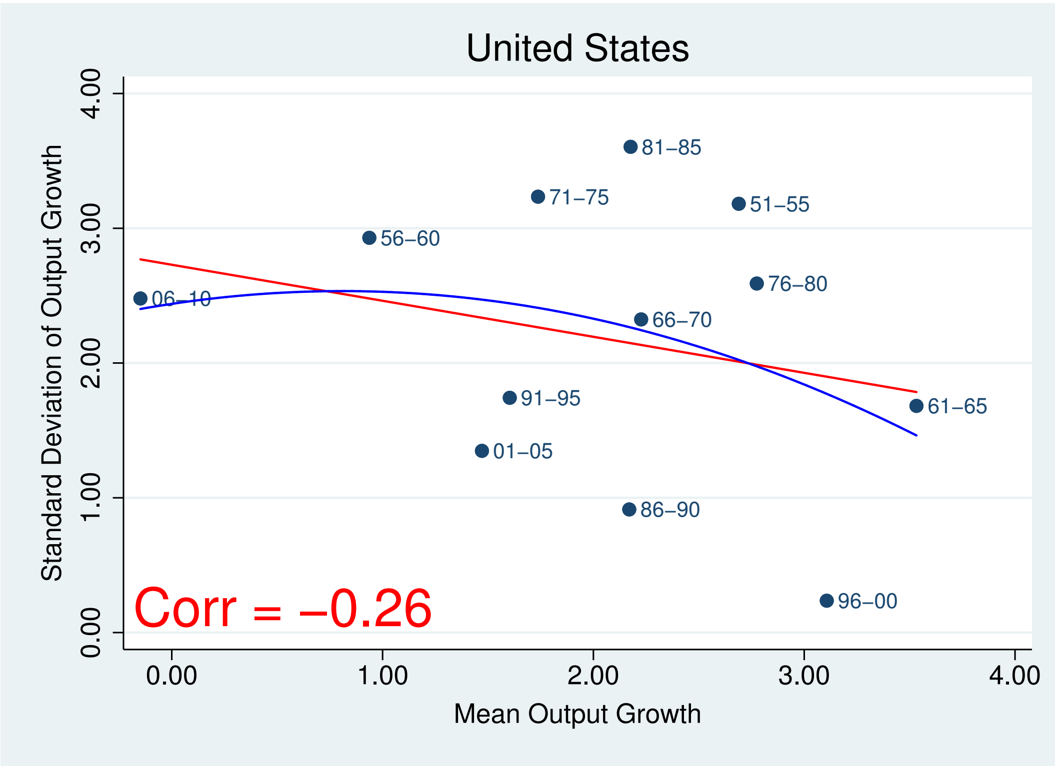

Figure 1.13 Growth and uncertainty: the United States.

1.2.3 OECD

Finally, let us analyze OECD. As of August 2019, OECD consists of 36 counties.[1] Unlike BRICS and OECD, it has a enough number of countries to perform a panel data analysis. So, we can produce scatter plots similar to those of Ramey and Ramey (1995), though they had the older data for only 24 OECD countries. As such, figures in the rest of Ch. 1 can be viewed as an updated version of their study. I fix the last year for 2014 to investigate as the latest data as possible, while varying the starting year. In this way, we can possibly capture the "period characteristic" such as Great Moderation since the mid-1980s.

Table 1.3: Correlation Coefficients and Their Significance: OECD

Period	Correlation Coefficient	p-value
1970 – 2014	+0.15	0.38
1980 – 2014	+0.24	0.15
1990 – 2014	+0.26	0.13
2000 – 2014	+0.48	0.00

The results are shown in Figs. 1.14 to 1.17 and as above, are summarized in Table 1.3. Though we basically observed *negative* links in a sample of BRICS and G7, we now see *positive* relations for all periods. In particular, for the period 2000−2014, a positive correlation is relatively strong and significant at 1%. As such, the finding of Ramey and Ramey (1995) has been *reversed*. This observation is consistent with a remark of Norrbin and Yigit (2005) that a negative link may be specific to OECD countries. We'll analyze the possible reason for this in theoretical parts below, but it is possibly due to a "structural change" since the mid-1980s (recall that the last year of Ramey and Ramey (1995) for an OECD sample was 1988).

Another feature of Figs. 1.14 to 1.17 is that a linear line basically coincides with a quadratic line; so nonlinearity is unlikely to be hidden in these figures. Put differently, the bottom line of our OECD analysis is that we see a significant, positive association for the latest period 2000−2014.

[1] See `http://www.oecd.org/about/members-and-partners/`. Colombia was invited to join and its accession is imminent; it'll be the OECD's 37th member country. The data for Colombia, however, are not included in my data set.

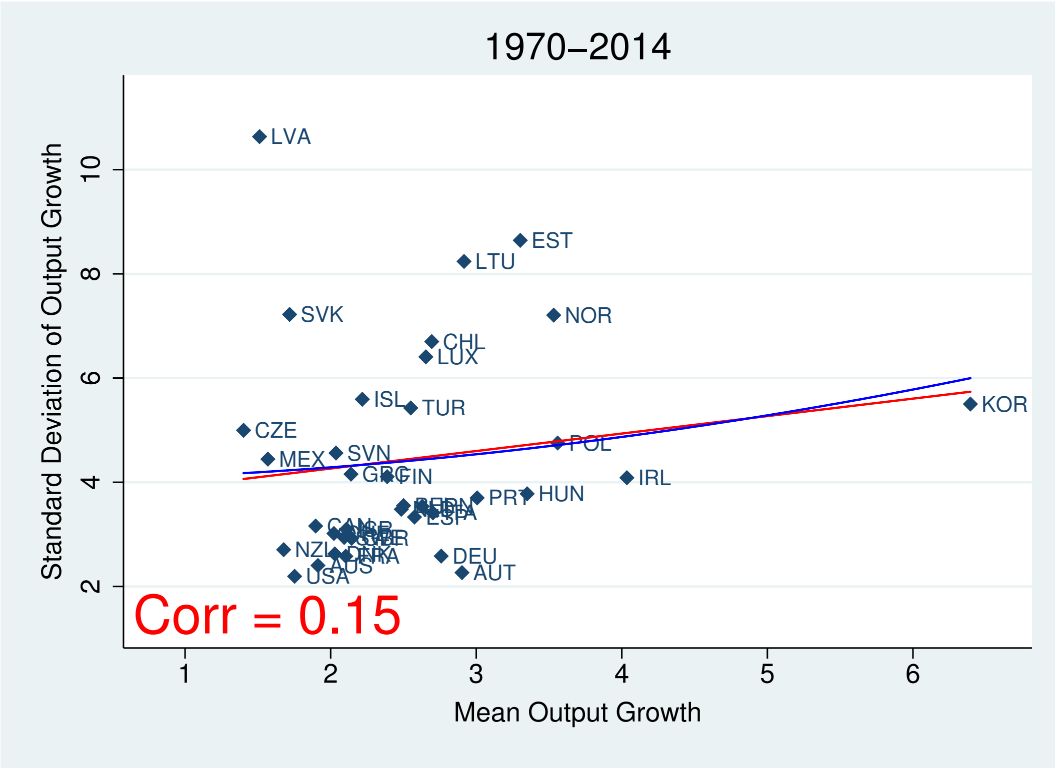

Figure 1.14 Growth and uncertainty in OECD: 1970-2014.

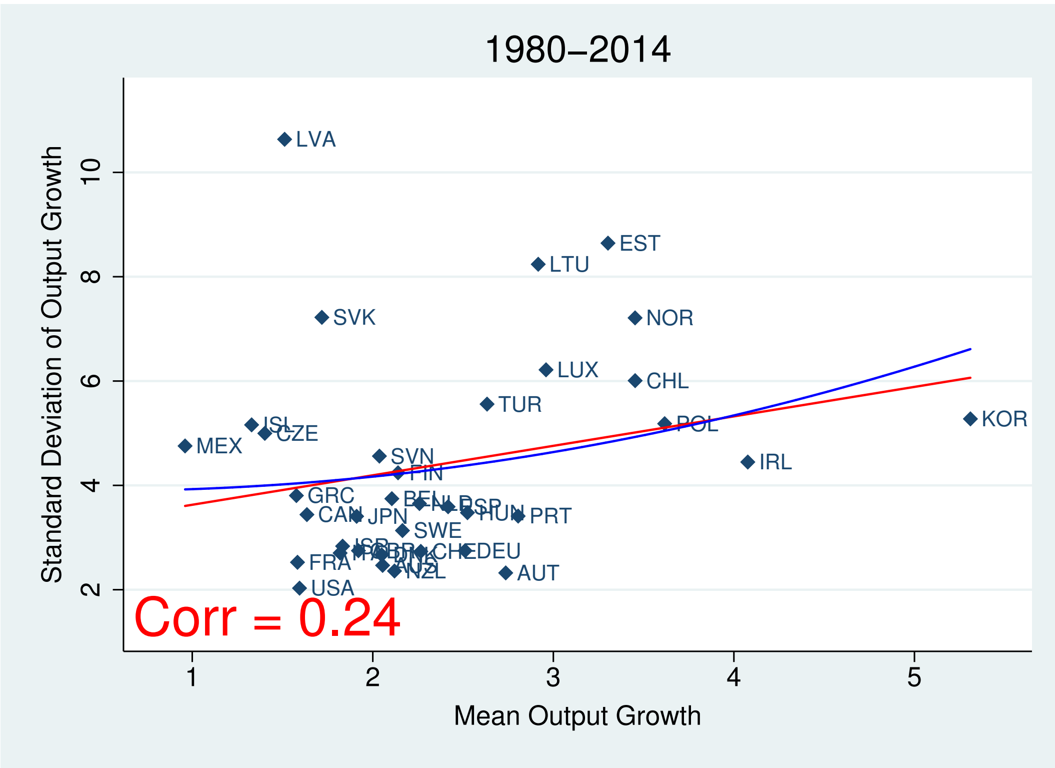

Figure 1.15 Growth and uncertainty in OECD: 1980-2014.

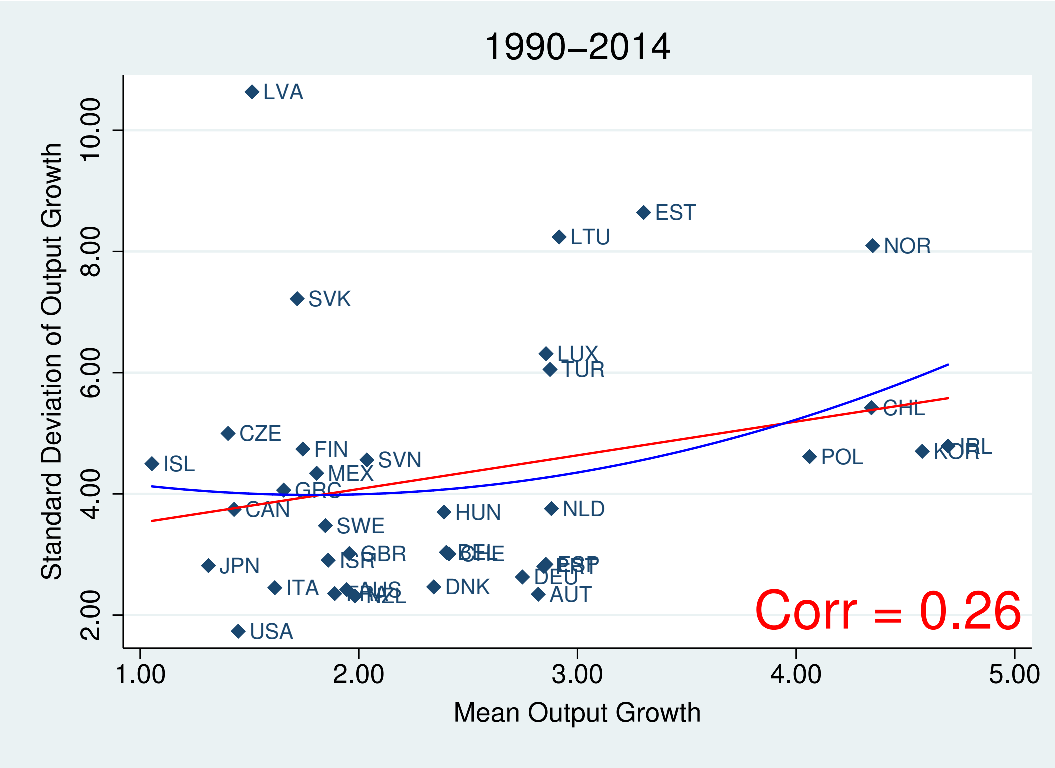

Figure 1.16 Growth and uncertainty in OECD: 1990-2014.

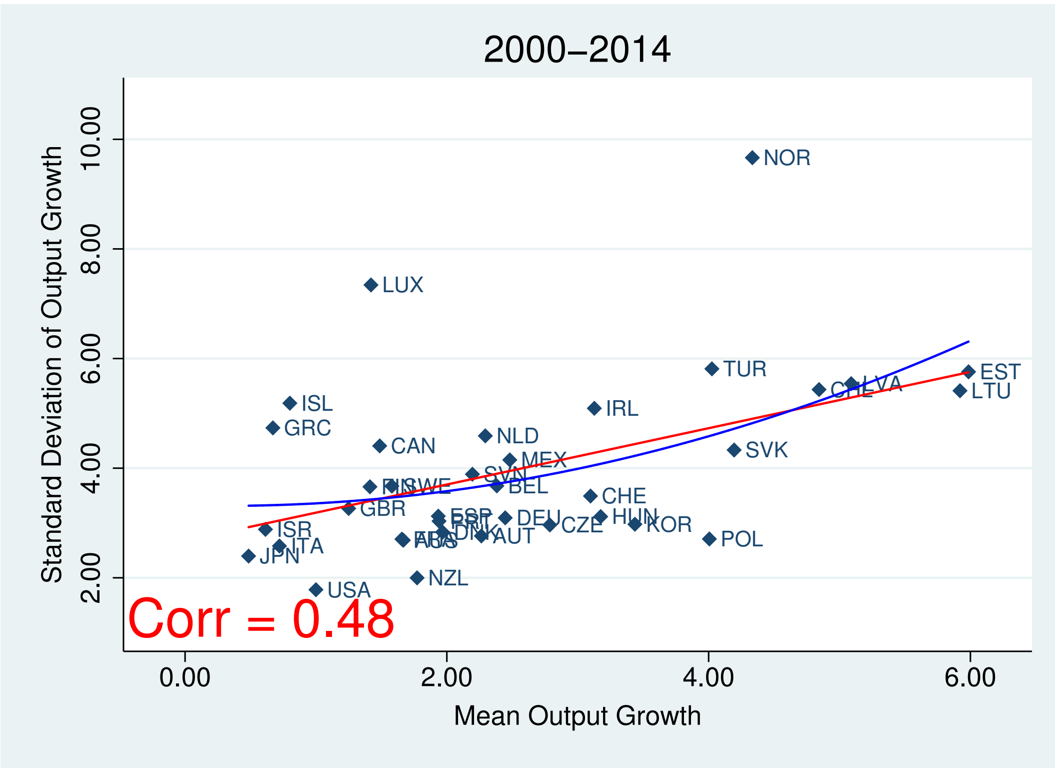

Figure 1.17 Growth and uncertainty in OECD: 2000-2014.

1.3 Input-level Analysis

So far, we have focused on "aggregate" uncertainty: uncertainty about *outputs*. Literature on economic growth, however, has discovered what *inputs* are crucial; physical capital, human capital, population, natural resources, and so on.[1] As growth is ultimately characterized by inputs, we need to look at "micro" uncertainty. Another reason for this is that empirical studies cited above have exclusively focused on aggregate uncertainty. As such, scatter plots below call for more systematic studies on the relationship between growth and micro uncertainty.

Of course, as we disaggregate, data are generally scarce, and we must resort to suitable proxies. For example, though years of schooling are often used in growth accounting, they don't describe the rate of growth of human capital well. Keeping this sort of difficulties in mind, this subsection investigates the relationship between growth and micro uncertainty about physical capital, population, and human capital.

1.3.1 Physical Capital

Let us start with physical capital. As a proxy, I use an investment-GDP ratio (I/Y). This is also from PWT 9.0. The standard deviation of I/Y may represent uncertainty about physical capital. I use the same technique to produce scatter plots as in the case of aggregate uncertainty for OECD countries above. The results are displayed in Figs. 1.18 to 1.21 and summarized in Table 1.4.

Table 1.4: Summary Statistics for OECD: Physical Capital

Period	**Correlation Coefficient**	p**-value**
1970 – 2014	+0.13	0.46
1980 – 2014	+0.05	0.78
1990 – 2014	−0.15	0.37
2000 – 2014	+0.04	0.84

In all figures, correlation coefficients are low and insignificant. Therefore, linear regression lines seem meaningless. At the same time, we see an inverted U-shaped

[1] In his recent survey on economic growth, Jones (2016, p.21) argues that we need to understand more about inputs (with special attention to misallocation). See Moll (2014), Mino (2015; 2016), and Nguyen (2019a; 2019b) for recent developments in the literature on capital misallocation and economic growth.

association in Fig. 1.18 and U-shaped curves in Figs. 1.20 and 1.21. Consequently, the link between growth and uncertainty about physical capital may be better captured by a nonlinear approximation than a linear one. Moreover, we see that a nonlinear pattern has changed over time. So, in estimating this relationship, a panel data approach is likely to produce a good outcome.

Summing up, a linear relationship between growth and physical capital uncertainty is insignificant for all periods. But we see an inverted U-shaped pattern in the first period $1970-2014$ and U-shaped patterns in the third $(1990-2014)$ and last $(2000-2014)$ periods.

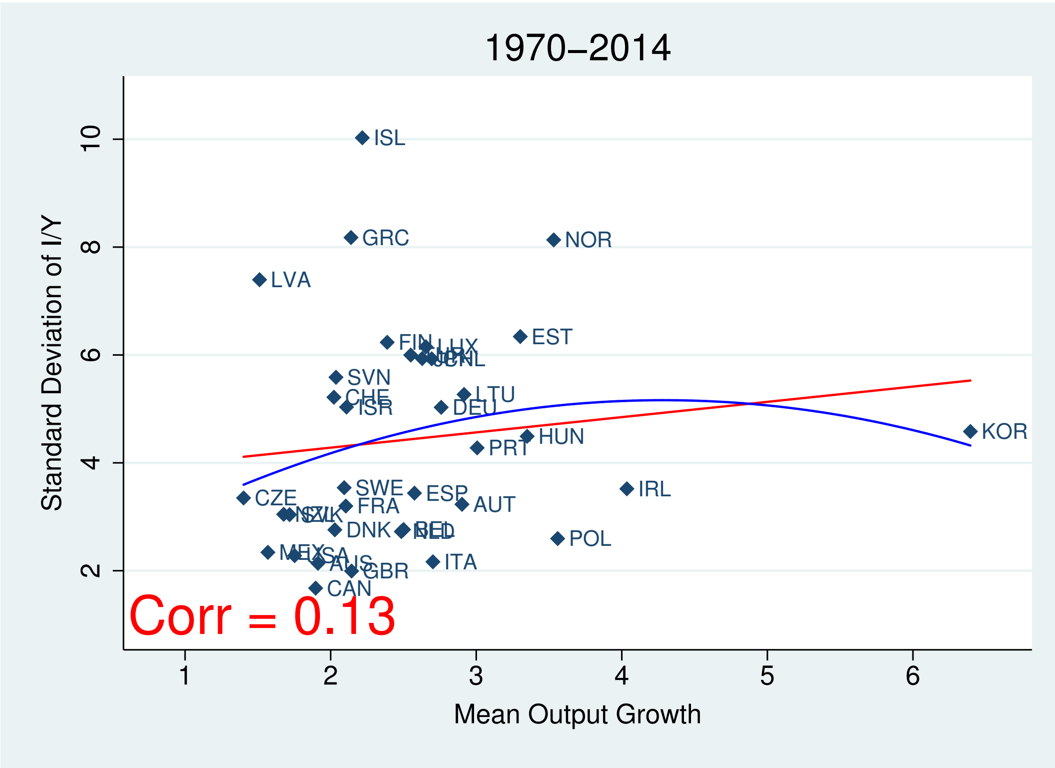

Figure 1.18 Growth and uncertainty about investment-to-GDP ratio in OECD: 1970-2014.

1.3.2 Population

Next, let us examine population. Unlike capital, we don't have to use a proxy; the data for population are again from PWT 9.0. Its standard deviation may represent, for instance, "changes in social mores and tastes with respect to child-bearing, natural disaster, wide-spread disease, discovery of a "wonder" drug, national eco-

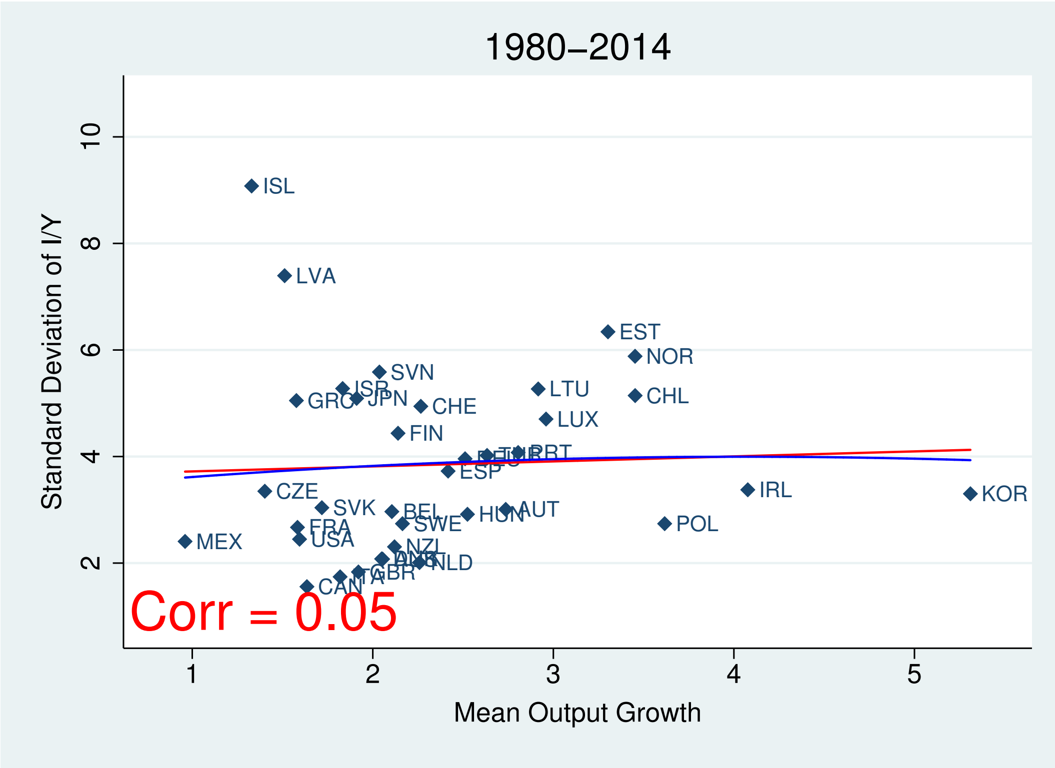

Figure 1.19 Growth and uncertainty about investment-to-GDP ratio in OECD: 1980-2014.

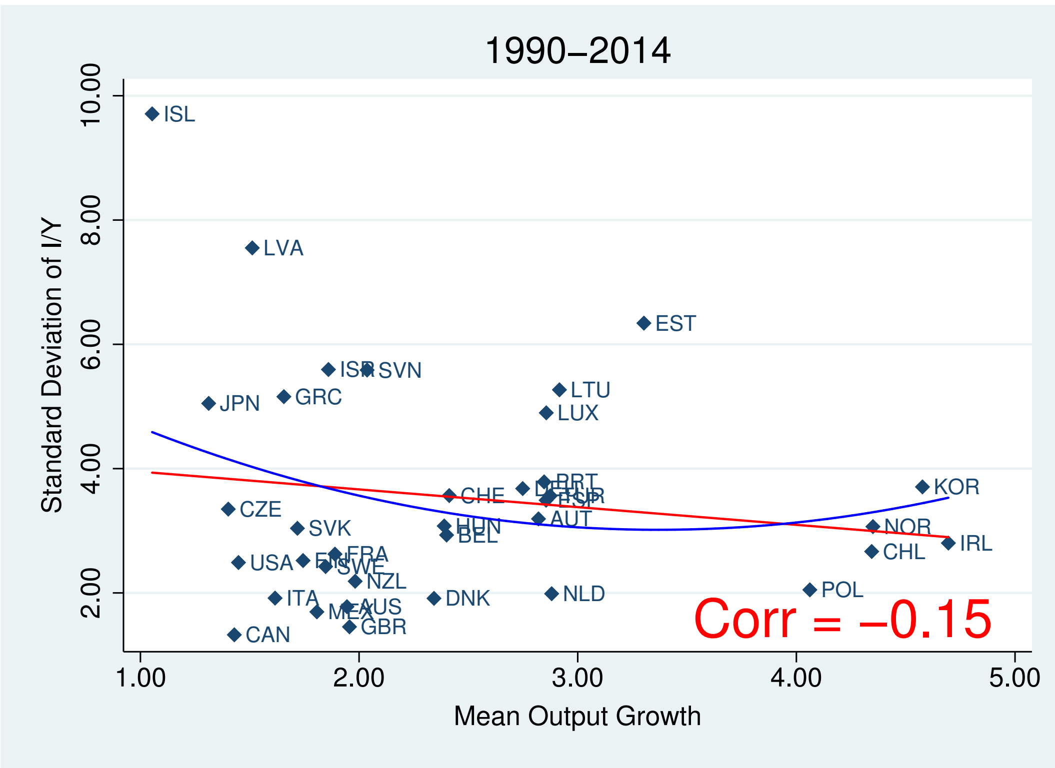

Figure 1.20 Growth and uncertainty about investment-to-GDP ratio in OECD: 1990-2014.

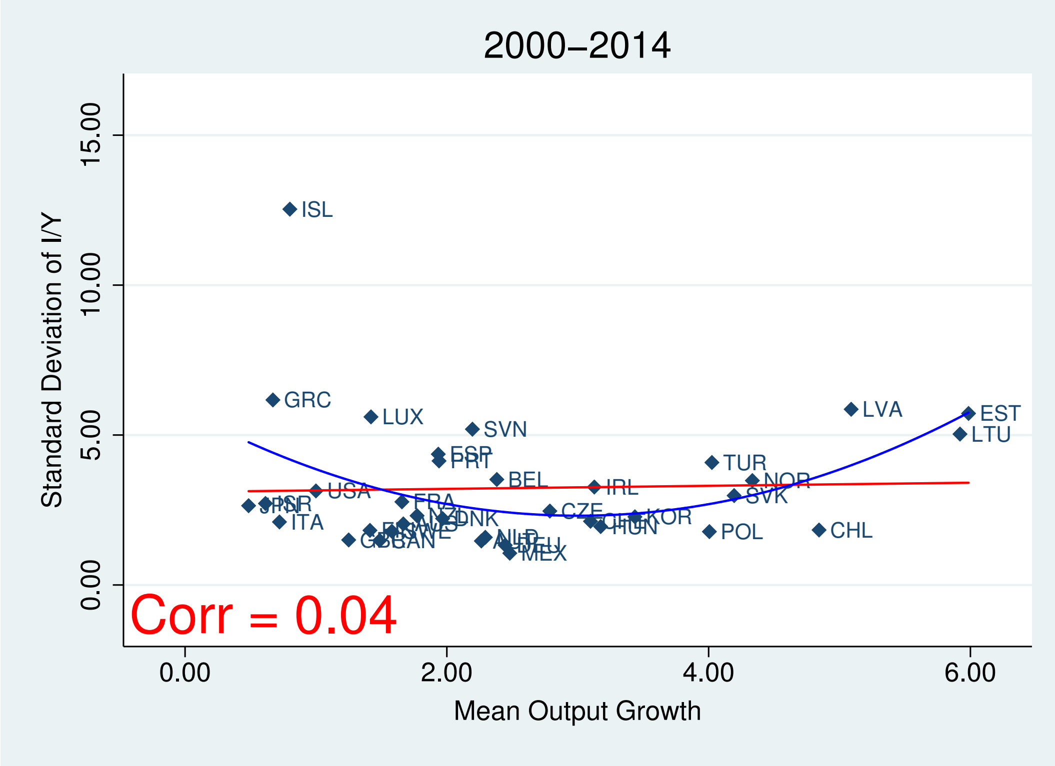

Figure 1.21 Growth and uncertainty about investment-to-GDP ratio in OECD: 2000-2014.

nomic conditions, etc." according to the seminal paper of Merton (1975, p.376). The results are shown in Figs. 1.22 to 1.25 and summarized in Table 1.5.

Table 1.5: Summary Statistics for OECD: Population

Period	Correlation Coefficient	p-value
1970 – 2014	+0.44	0.01
1980 – 2014	+0.44	0.01
1990 – 2014	+0.26	0.12
2000 – 2014	−0.09	0.61

Unlike physical capital, we see some significant results. For example, a *positive*, modestly strong association for the first (1970 – 2014) and second (1980 – 2014) periods is significant at 1%. This positive link contrasts with a negative one of Ramey and Ramey (1995); thus, as long as one focuses on aggregate uncertainty, it is impossible to dig out this micro relation. The disappearance of a significant link since 1990 may suggest (exogenous) changes in demographic patterns among 36 OECD countries.

In terms of linearity, we virtually observe the coincidence of linear and quadratic lines for all periods. Therefore, the relationship between growth and demographic uncertainty seems linear. To sum up, we see positive, significant correlations when earlier periods 1970s and 1980s are included.

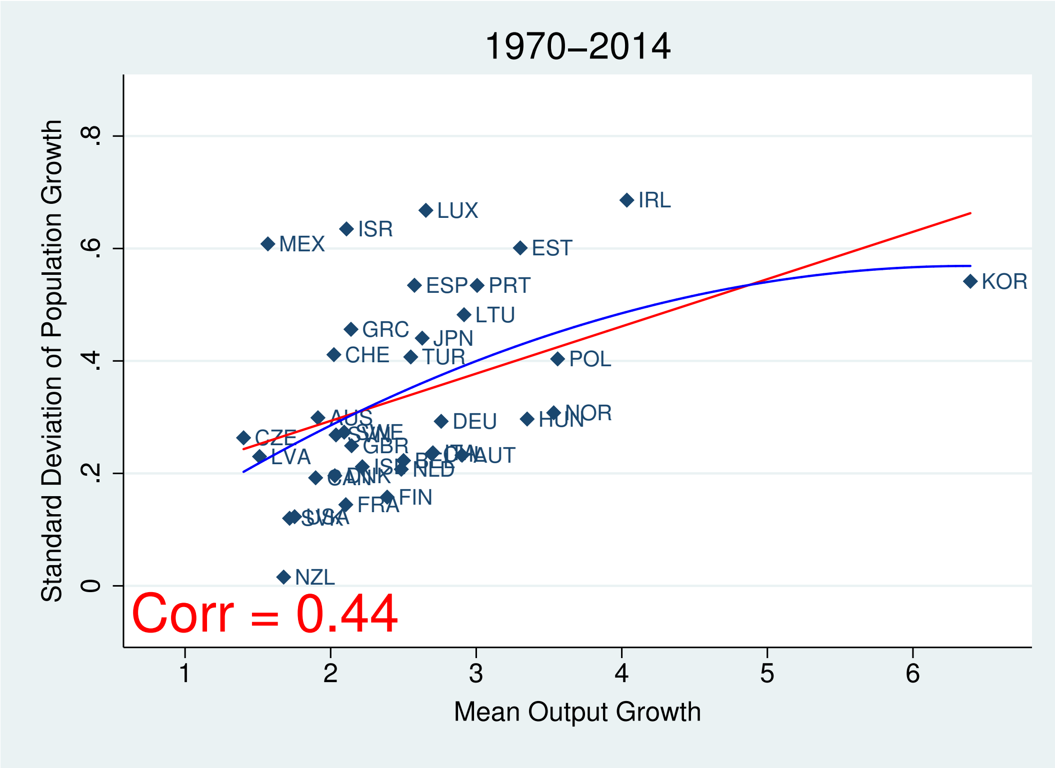

Figure 1.22 Growth and demographic uncertainty in OECD: 1970-2014.

1.3.3 Human Capital

Finally, let us consider human capital. Obviously, as slightly mentioned above, finding the long-run data for human capital is considerably difficult. For this reason, I use the data on health spending as a proxy.[1] Ideally, we want to look at the data on, say, public spending on education (as a percentage of GDP). Those data, however, are short-run and have the few degrees of freedom.[2] Instead, we can think of it in this way: if children are sick, they cannot go to school; if adults

[1] They are from OECD Data (`https://data.oecd.org/healthres/health-spending.htm`) and begin with the year 1970. It is described as follows: "Health spending measures the final consumption of health care goods and services (i.e. current health expenditure) including personal health care (curative care, ancillary services and medical goods) and collective services (prevention and public health services as well as health administration), but excluding spending on investment."

[2] For example, the data on public spending from OECD Data begin only with the year 1995.

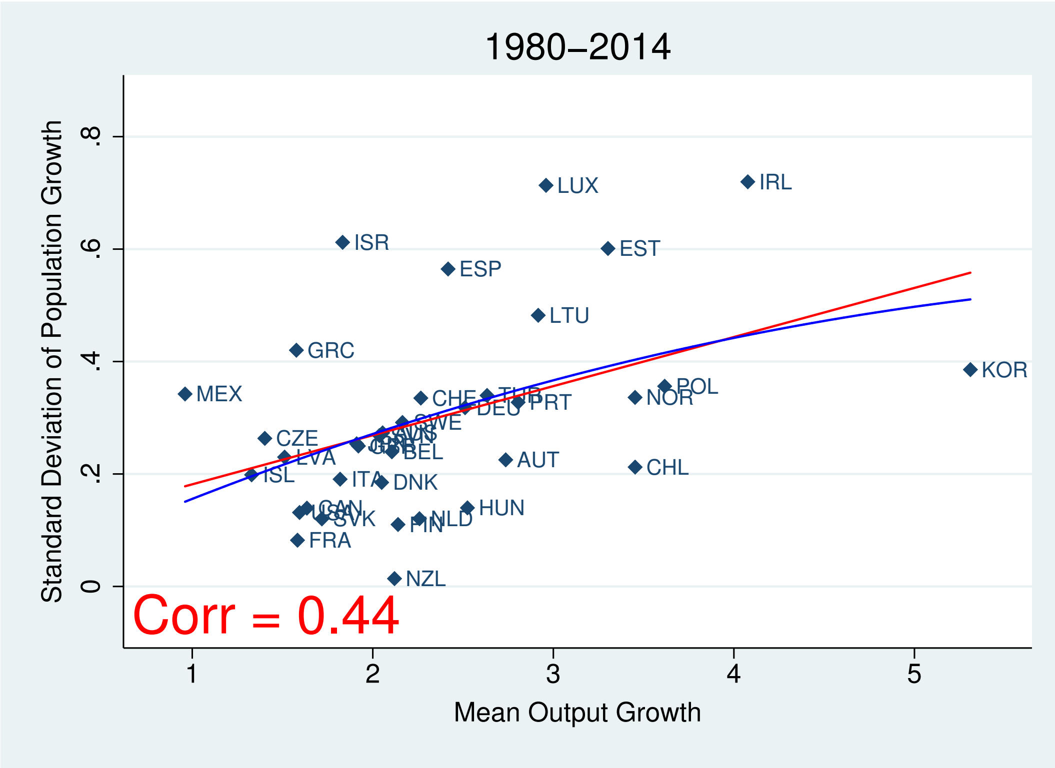

Figure 1.23 Growth and demographic uncertainty in OECD: 1980-2014.

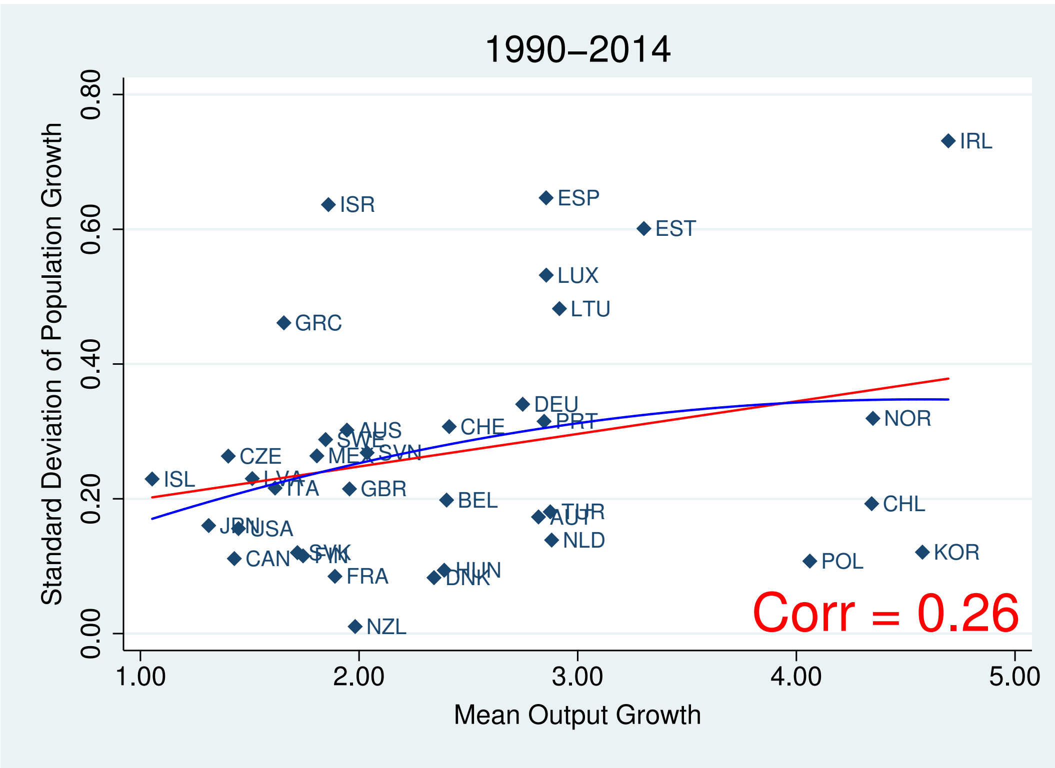

Figure 1.24 Growth and demographic uncertainty in OECD: 1990-2014.

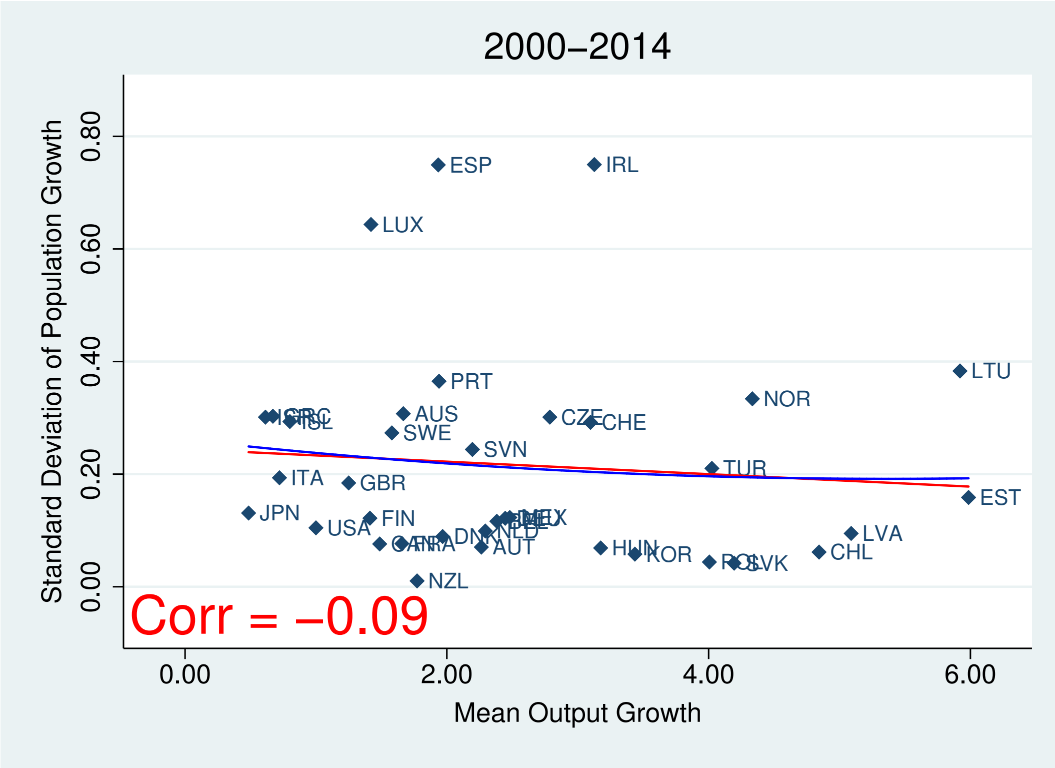

Figure 1.25 Growth and demographic uncertainty in OECD: 2000-2014.

are sick, they cannot participate in the on-the-job training (OJT). In this sense, health is the necessary condition for receiving education or obtaining skills, that is, the accumulation of human capital. Therefore, we expect that health status and educational attainment are strongly, positively correlated.

Table 1.6: Summary Statistics for OECD: Human Capital

Period	**Correlation Coefficient**	***p*-value**
1970 – 2014	−0.01	0.96
1980 – 2014	±0.00	0.99
1990 – 2014	+0.03	0.87
2000 – 2014	−0.28	0.09

The results are shown in Figs. 1.26 to 1.29 and summarized in Table 1.6. In large part, they are unfavorable; for the earlier periods (1970s, 1980s, and 1990s), correlation coefficients are virtually zero with a fairy high p-value. What is worse, while we see a U-shaped link for the period 1990 – 2014, linear and quadratic approximations coincide for the earlier periods.

For the latest period 2000 – 2014, however, we see a significant (at 10%), *negative* association between growth and human capital uncertainty among 36

OECD economies. In sum, we see a U-shaped link for the third $(1990-2014)$ period, and a negative relation for the latest period. This completes our empirical inquiry into the growth-uncertainty nexus, both at the aggregate and micro level.

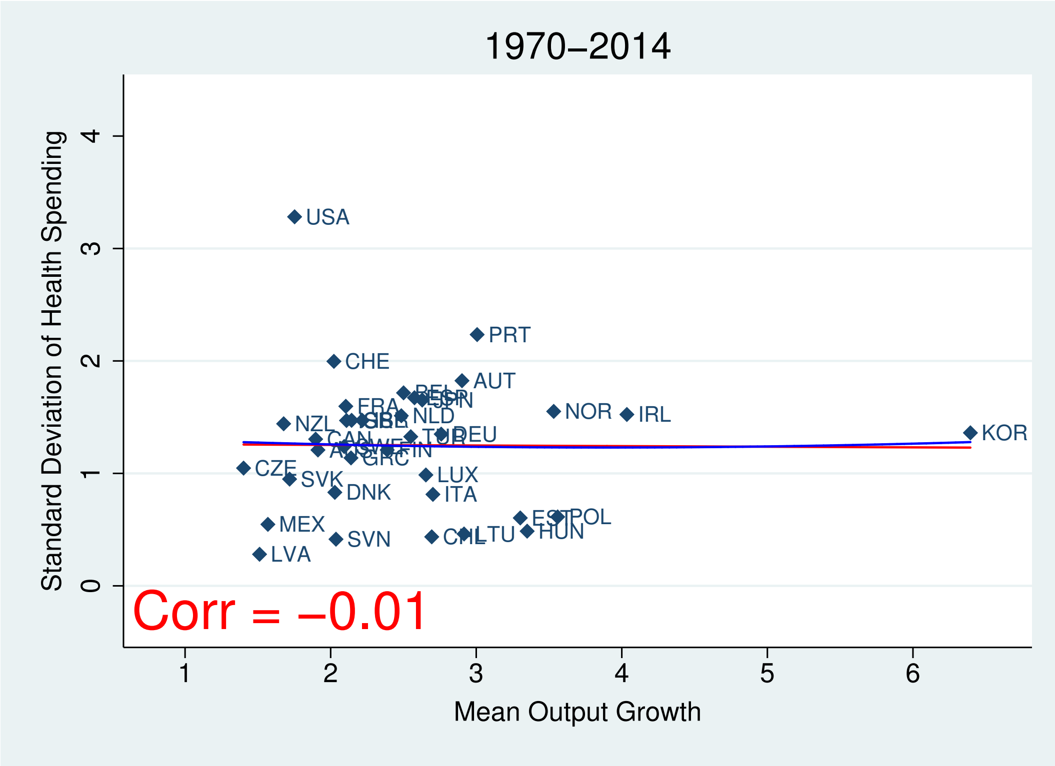

Figure 1.26 Growth and human capital uncertainty in OECD: 1970-2014.

1.4 Organization

Having reviewed related empirical studies and looked at a large sets of available evidence on the growth-uncertainty nexus, we are now ready to undertake a theoretical analysis. The rest of this thesis consists of five chapters; four for a theoretical analysis and one for a conclusion. Four theoretical chapters draw heavily on research that I have undertaken at the end of the Heisei era. At the appropriate places in each chapter, I have indicated the original source of research from which the presentation has been adapted. In all cases, however, the material has been extensively revised; the data used in the figures have been extended or replaced by the better data set; improvements to the exposition have been made in all chapters to make the thesis more readable and accessible. I hope that a

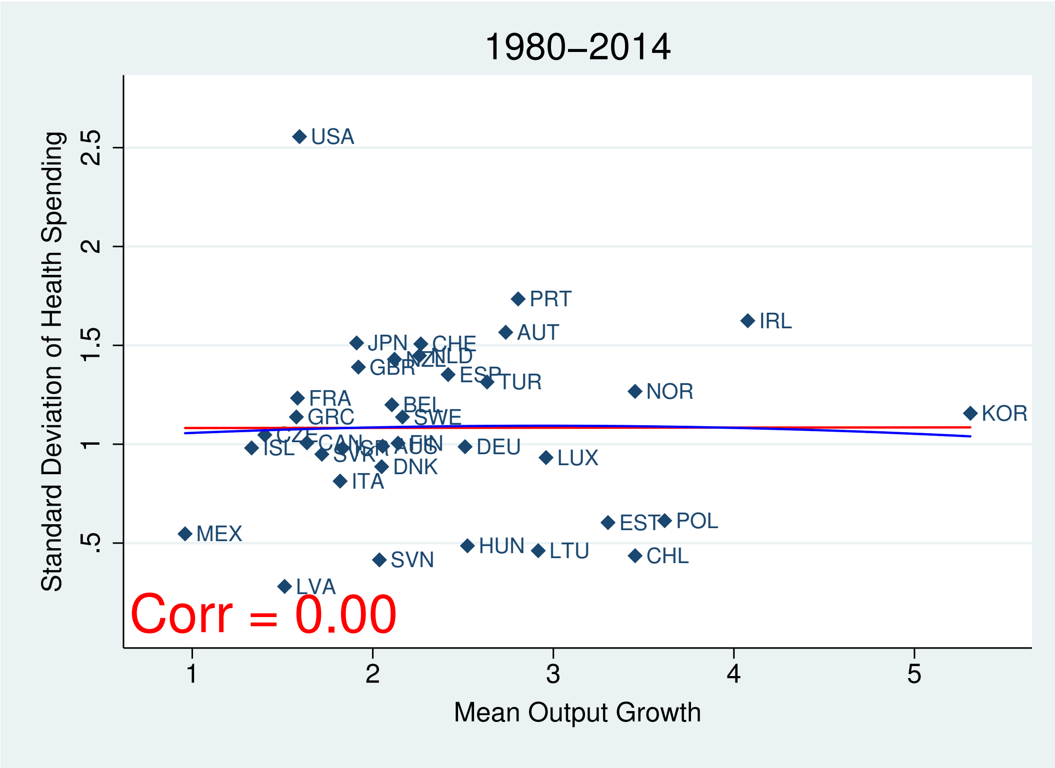

Figure 1.27 Growth and human capital uncertainty in OECD: 1980-2014.

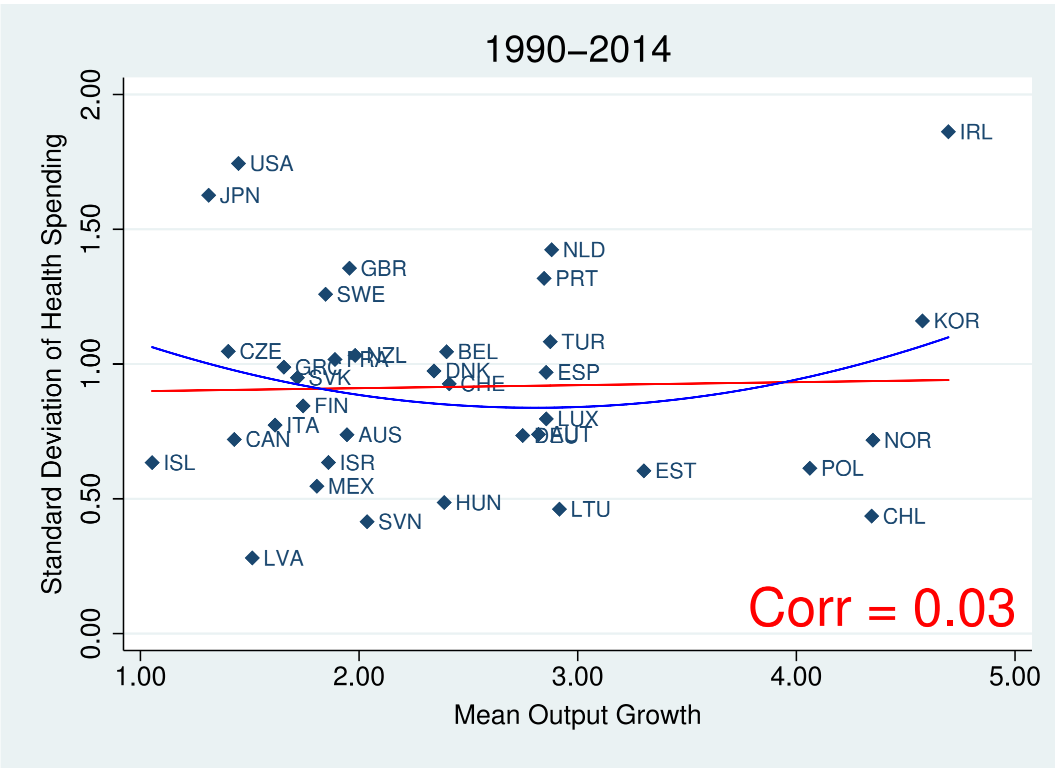

Figure 1.28 Growth and human capital uncertainty in OECD: 1990-2014.

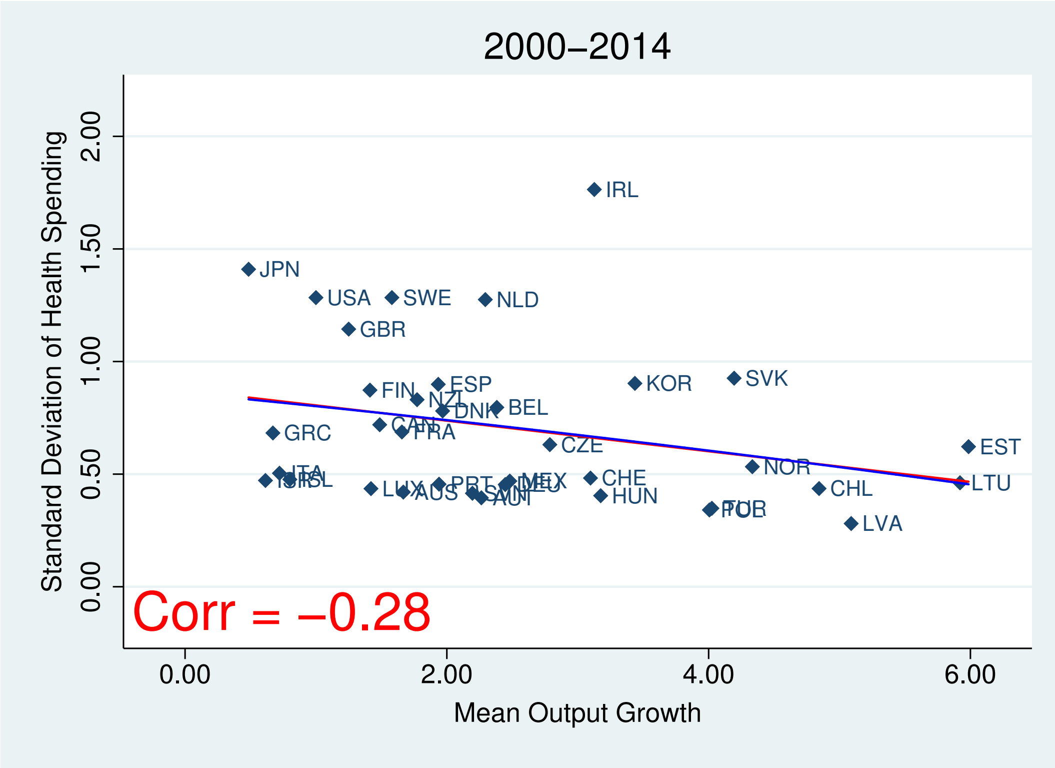

Figure 1.29 Growth and human capital uncertainty in OECD: 2000-2014.

reader finds the chapters in this thesis much different from the original published version. All chapters are entirely self-contained; thus, you can read this thesis in any order.

Specifically, Ch. 2 develops a baseline model we will use throughout this thesis: the Uzawa-Lucas growth model *under uncertainty*. Originally constructed by Uzawa (1965), it features so-called *endogenous growth* based on the accumulation of human capital. When the early 1980s witnessed new developments in the theory of imperfect competition, Lucas (1988) slightly elaborates on Uzawa (1965), hence called the Uzawa-Lucas model. To account for some of correlates we saw above, however, I will incorporate uncertainty into the baseline model. By "baseline," I mean the absence of *correlations* between stochastic processes. Models in subsequent chapters extend this baseline model by assuming some sort of correlation between them. Ch. 2 also contains technical materials such as the proof of stochastic transversality condition, the derivation of a solution to the stochastic differential equation, etc.

Ch. 3 is the first place where I relax the assumption of uncorrelated stochastic processes. It presents the extended model in which stochastic resource dynamics and stochastic technological progress are correlated. In the benchmark case with no correlation, I will find a negative relation. When there is a positive correlation, however, I will find an inverted U-shaped relationship between growth and uncertainty.

Until Ch. 4, the role of population dynamics is assumed away. This chapter explores a link between population dynamics and technological progress under uncertainty. Here, in the benchmark uncorrelated case, there exists a positive association between growth and uncertainty. In contrast, when there is a negative correlation, there is a U-shaped relationship between growth and uncertainty. So, at this stage, all four patterns are theoretically replicated, but separately.

Thus, the final Ch. 5 is devoted to the development of an extended model in which *all* four patterns identified by empirical studies can be replicated in *one* place. The model abstracts from technological progress. In that sense, it is simpler than the baseline model. It, however, considers a correlation between the accumulation of physical capital and population dynamics, and a correlation between the accumulation of human capital and population dynamics. I will show that all four patterns emerge according to a variation in a correlation parameter.

Concluding remarks appear in Ch. 6. As results are summarized in each chapter, instead of repeating them, I will outline limitations of this thesis and possible extensions.

Finally, Appendix A presents a *deterministic* Uzawa-Lucas growth model. In the absence of uncertainty, a characterization of the steady state is analytically straightforward, and the model's main mechanisms are easier to grasp by analyzing a steady state. Therefore, this Appendix is for a reader either who is new to a Uzawa-Lucas model or who wishes to refresh your memory. I note, however, that I have put lots of efforts in writing this Appendix so that even a reader familiar with this model may learn something new.

Chapter 2

The Baseline Model

2.1 Introduction

This is the first chapter that deals with a *stochastic* Uzawa-Lucas model.[1] It solves the model and explains its major economic properties. It extends the deterministic version presented in Appendix A. Specifically, following Krebs (2003) and Hiraguchi (2018), I extend Hiraguchi (2013) by assuming a stochastic accumulation of human capital.

This chapter is organized as follows. Sect 2.2 solves the stochastic Uzawa-Lucas model in which human capital accumulation is driven by a Brownian motion process. In Sect. 2.3, I present a more general version with the combination of a Brownian motion process and many Poisson jump processes. In Sect. 2.4, I consider the stochastic accumulation of *physical* capital as well. Concluding remarks appear in Sect. 2.5.

Technical materials are all relegated to Appendix. Appendix 2.A describes how to guess the functional form of a value function. Appendix 2.B proves the transversality condition under uncertainty. Appendix 2.C shows how to solve a standard stochastic differential equation. Appendix 2.D reviews the literature on stochastic growth models that have tried finding their closed-form solutions.

[1] This chapter is based on Tsuboi (2018).

2.2 The Model

In this section, I construct a stochastic Uzawa-Lucas model in which the accumulation of human capital follows a Brownian motion process. Following Bucci et al. (2011) and Hiraguchi (2013), I normalize the total number of workers L equals unity ($L = 1$) to simplify our analysis. So, per capita terms are equivalent to aggregate terms. We'll relax this assumption in later chapters.

A representative household is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u(t) \in (0, 1)$ denote the fraction of time spent working to produce final goods $Y(t)$. Correspondingly, $1 - u(t)$ represents the fraction of time spent learning to accumulate new human capital. The amount of leisure is fixed exogenously, so there is no choice about it.[1]

2.2.1 Capital Accumulation and Household

The accumulation of human capital $H(t)$ is stochastically governed by the following rule

$$dH(t) = b(1 - u(t))H(t)dt - \delta_H H(t)dt + \sigma_H H(t)dz_H(t), \tag{2.1}$$

where $b > 0$ is an exogenous parameter that indicates how efficient human capital accumulation is. $\delta_H \in (0, 1)$ is its depreciation. $dz_H(t)$ is the increment of a Brownian motion process such that the mean $\mathcal{E}(dz_H) = 0$ and variance $\mathcal{V}(dz_H) = dt$, and $\sigma_H \geq 0$ is the associated diffusion coefficient of human capital (if $\sigma_H = 0$, then we would recover a deterministic limit). The initial stock of human capital $H(0) = H_0 > 0$ is given, so that $H(t) > 0$ for all t with probability 1.

Note that a stochastic process (2.1) is a controlled diffusion process; that is, it contains one of key control variables in a Uzawa-Lucas model, $u(t)$. Bucci et al. (2011) and Hiraguchi (2013) assume that technological progress is stochastic, while human capital accumulation is deterministic ($\sigma_H = 0$). This is at odds

[1] As an explicit incorporation of leisure precludes an analytical solution, I abstract from it. See Benhabib and Perli (1994), Ladrón-De-Guevara et al. (1999), and Solow (2000) for the deterministic Uzawa-Lucas model with leisure. No study has found the closed-form solution to the stochastic Uzawa-Lucas model with leisure.

with the empirical literature such as Hartog et al. (2007).[1] Hartog et al. (2007) construct a simulation model to replicate the situation in which agents *ex ante* face risks associated with education. They empirically demonstrate that investment in a college education is as risky as investment in the stock market with a portfolio of some 30 randomly chosen stocks; hence *stochastic* returns from human capital accumulation. Bilkic et al. (2012) also examine human capital uncertainty by evaluating the decision of students on when to leave school and to enter the labor market.

The economy-wide resource constraint is governed in a deterministic way:

$$dK(t) = \underbrace{(u(t)H(t))^{\alpha}K(t)^{\beta}A(t)^{\gamma}}_{\equiv Y(t)}dt - \delta_K K(t)dt - C(t)dt, \tag{2.2}$$

where $\gamma = 1 - \alpha - \beta$. $K(t)$ is physical capital, and $\delta_K \in (0,1)$ is its depreciation rate. $\alpha \in (0,1)$ is the human capital share of income in a Cobb-Douglas production function. $C(t)$ denotes consumption of final goods. The initial stock of physical capital $K(0) = K_0 > 0$ is also given. We will examine the stochastic version of (2.2) in Sect. 4.

$A(t)$ is technology. Its law of motion is simply

$$dA(t) = \mu A(t)dt, \tag{2.3}$$

where $\mu > 0$. This is stochastically modelled in Bucci et al (2011) and Hiraguchi (2013). As the focus of this chapter is on the stochastic accumulation of human capital, I keep (2.3) deterministic throughout. The initial stock of technology $A(0) = A_0 > 0$ is given as well.

Finally, preferences of a representative household are given by the standard constant relative risk averison (CRRA) utility function:

$$E\int_0^{\infty} e^{-\rho t}\frac{C(t)^{1-\phi} - 1}{1-\phi}dt, \tag{2.4}$$

where E is the mathematical expectation operator with respect to the information set available to a representative household. $\rho > 0$ is a subjective discount rate;

[1] The lack of human capital uncertainty is also pointed out by Levhari and Weiss (1974) and Krebs (2003).

that is, the rate at which utility is discounted. $\phi > 0$ is the index of relative risk aversion (and $1/\phi$ is an elasticity of intertemporal substitution). When future consumption is uncertain, a larger ϕ makes future utility gain smaller, raising the value of additional future consumption. A representative household maximizes its expected utility (2.4) subject to a stochastic process (2.1) and the law of motion for physical capital (2.2) and technological progress (2.3).

2.2.2 Optimization

In order to solve this optimization problem, let $J(K, H, A)$ be a value function (or an indirect utility function). Then, the associated Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\begin{aligned} \rho J(K,H,A) &= \max_{\{C_t,u_t\}} \left(\frac{C(t)^{1-\phi}-1}{1-\phi} + J_K \frac{dK}{dt} + J_H \frac{dH}{dt} + J_A \frac{dA}{dt} + \frac{J_{H_H}}{2} \frac{(dH)^2}{dt} \right) \\ &= \max_{\{C,u\}} \left(\frac{C^{1-\phi}-1}{1-\phi} + J_K (uH)^{\alpha} K^{1-\alpha} - J_K \delta_K K - J_K C + J_A \mu A \right. \\ &\left. + J_H b(1-u)H - J_H \delta_H H + \frac{J_{HH} H^2 \sigma_H^2}{2} \right) \end{aligned} \tag{2.5}$$

where $J_K \equiv \partial J/\partial K$, $J_H \equiv \partial J/\partial H$, $J_A \equiv \partial J/\partial A$ and $J_{H_H} \equiv \partial^2 J/\partial H^2$. First-order conditions with respect to C and u are respectively

$$C = J_K^{-\frac{1}{\phi}}, \tag{2.6}$$

and

$$u = \left(\frac{\alpha J_K}{b J_H} \right)^{\frac{1}{1-\alpha}} \frac{A^{\frac{\gamma}{1-\alpha}} K^{\frac{\beta}{1-\alpha}}}{H}. \tag{2.7}$$

Substituting first-order conditions (2.6) and (2.7) back to a HJB equation (2.5), and rearranging, we get the maximized HJB equation of the form

$$\rho J(K,H,A) = \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha}\right) A^{\frac{\gamma}{1-\alpha}} K^{\frac{\beta}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}}$$
$$- J_K \delta_K K + J_A \mu A + J_H b H - J_H \delta_H H + \frac{J_{HH} H^2 \sigma_H^2}{2}.$$

Note that this is a partial differential equation. In general, it is impossible to solve it analytically. Nonetheless, we can find a closed-form solution with one parameter restriction. It can be summarized as follows:

Theorem 2.1. *When*

$$\phi = \beta, \tag{2.8}$$

there exists the closed-form representation of a value function (that satisfies the transversality condition, or TVC)

$$J(K,H,A) = \mathbb{X} K^{\alpha+\gamma} + \mathbb{Y} H^{\alpha} A^{\gamma} + \mathbb{Z}, \tag{2.9}$$

where

$$\mathbb{X} \equiv \frac{1}{\alpha+\gamma} \left(\frac{\beta}{\rho + (\alpha+\gamma)\delta_K}\right)^{\beta},$$

$$\mathbb{Y} \equiv \frac{1}{b^{\alpha}} \left(\frac{\beta}{\rho + (\alpha+\gamma)\delta_K}\right)^{\beta} \left(\frac{1-\alpha}{\rho - \mu\gamma - \alpha b + \alpha\delta_H + \frac{\sigma_H^2}{2}\alpha(1-\alpha)}\right)^{1-\alpha}, \tag{2.10}$$

and

$$\mathbb{Z} \equiv -\frac{1}{\rho(\alpha+\gamma)}.$$

Moreover, control variables are expressed as

$$C = \frac{\rho + (\alpha+\gamma)\delta_K}{\beta} K, \tag{2.11}$$

and

$$u = \frac{\rho - \mu\gamma - \alpha b + \alpha\delta_H + \frac{\sigma_H^2}{2}\alpha(1-\alpha)}{b(1-\alpha)}. \tag{2.12}$$

Proof. See Appendix 2.A.[1]

2.2.3 Macroeconomic Implications

I in turn comment on main points in Theorem 1.

2.2.3.1 Parameter Restriction and Value Function

The parameter restriction (2.8) says that the risk aversion parameter equals the physical capital share of income. It allows us to find out the closed-form representation of a value function (2.9). Whether it holds true in practice is still open debate, because the estimate of ϕ is a task of great difficulty.[2] This restriction, however, has been widely used by a number of authors in order to obtain the closed-form solution to their model. Xie (1991, 1994), Rebelo and Xie (1999), Smith (2007), Bucci et al (2011), Marsiglio and La Torre (2012a; b), and Hiraguchi (2013; 2014) all use a restriction (2.8) to generate the insights that cannot be appreciated without an explicit solution. Following them, I also use (2.8).

Here, we can see that physical capital K and the product of technology and human capital AH are separable. This is in sharp contrast to Bucci et al (2011); they find these three state variables are *all* separable. As Hiraguchi (2013, p.137) notes, the economic implication of *non*separability between A and H is that the long-run engine of stochastic endogenous growth models is a fusion of technology and human capital. This is consistent with recent empirical studies such as Madsen (2014) and Cinnirella and Streb (2017): they emphasize the importance of the *interaction* between technological progress and human capital accumulation.

[1] We can be sure that $u \in (0,1)$ as long as the inequality

$$\alpha(b-\delta_H) - \frac{\sigma_H^2}{2}\alpha(1-\alpha) + \mu\gamma < \rho < b - \alpha\delta_H - \frac{\sigma_H^2}{2}\alpha(1-\alpha) + \mu\gamma$$

holds. The proof of TVC is in Appendix 2.B. As you will see, it is mathematically involved; thus, I present the proof of TVC only for Theorem 1 of this chapter.

[2] For example, on the one hand, Lucas (2003) claims that ϕ ranges from 1 (log utility) to 4, but on the other, Smith (2007) says that ϕ should be smaller than 1.

Moreover, I use a value function (2.9) for our welfare analysis. As Turnovsky (1997; 2000) show, when a value function is expressed in an explicit form, we can use it to assess the effect of parameters or variables of interest on welfare, in particular the impact of uncertainty on welfare. It is only possible in stochastic growth models, because deterministic models have, by construction, nothing to say about the effect of uncertainty on welfare.

2.2.3.2 Control Variables and Expected Growth Rate

Eq. (2.11) tells us that a consumption-capital ratio is constant. It might be at odds that the optimal level of consumption C depends neither on human capital stock H nor technology A. Nevertheless, on this point, Hiraguchi (2013, p.137) succinctly puts as follows: "We cannot not find the intuitive explanation why the current consumption level c is independent of the TFP level A and the human capital H. However, these values affect the physical capital accumulation and then they affect the *future* consumption. The independence result is consistent with Smith (2007) who obtains the closed-form solution to the one-sector neoclassical growth model." This property is also documented in Wälde (2011a)'s survey on one-sector stochastic growth models.

Another (seemingly) unpleasant property is that a consumption-capital ratio is irrelevant to the shock term σ_H.[1] This point is indicated by a horizontal dashed line in Fig. 2.1. In what follows, I use the following "standard" parameter values: $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\rho = 0.05$, and $\delta_K = \delta_H = 0.03$.[2] This parameterization doesn't violate the inequality guaranteeing $u \in (0, 1)$; so it illustrates a model's empirical validity to some extent. In Fig. 2.1, we see that consumption-capital ratio C/K is independent of how large human capital uncertainty σ_H is. Do human capital, technology, and demographic shocks have nothing to do with

[1] MATLAB codes for Figs. 2.1 and 2.3 are at `https://link.springer.com/article/10.1007/s00712-018-0604-6`

[2] Following the seminal paper of Mankiw et al (1992, p.432), I set the human capital share $\alpha = 1/3$. For physical capital share, it has been commonplace in macroeconomics to assume $\beta = 1/3$. As Karabarbounis and Neiman (2014) document, however, a labor share is declining globally. Therefore, I set $\gamma = 0.27$ so that the physical capital share roughly equals 0.40, the value used by Ahn et al (2018). $b = 0.11$ is used when Barro and Sara-i-Martin (2004) simulate a Uzawa-Lucas model. I choose $\mu = 0.02$ and $\delta_K = \delta_H = 0.03$, again following Mankiw et al (1992). Finally, following Caballé and Santos (1993) and Moll (2014), I set $\rho = 0.05$. In Fig. 2.1, I use $K = 10$ to make it transparent.

the optimal level of consumption? This is the important point missed in Hiraguchi (2013) and others, and therefore we will explore this in Sect. 4.

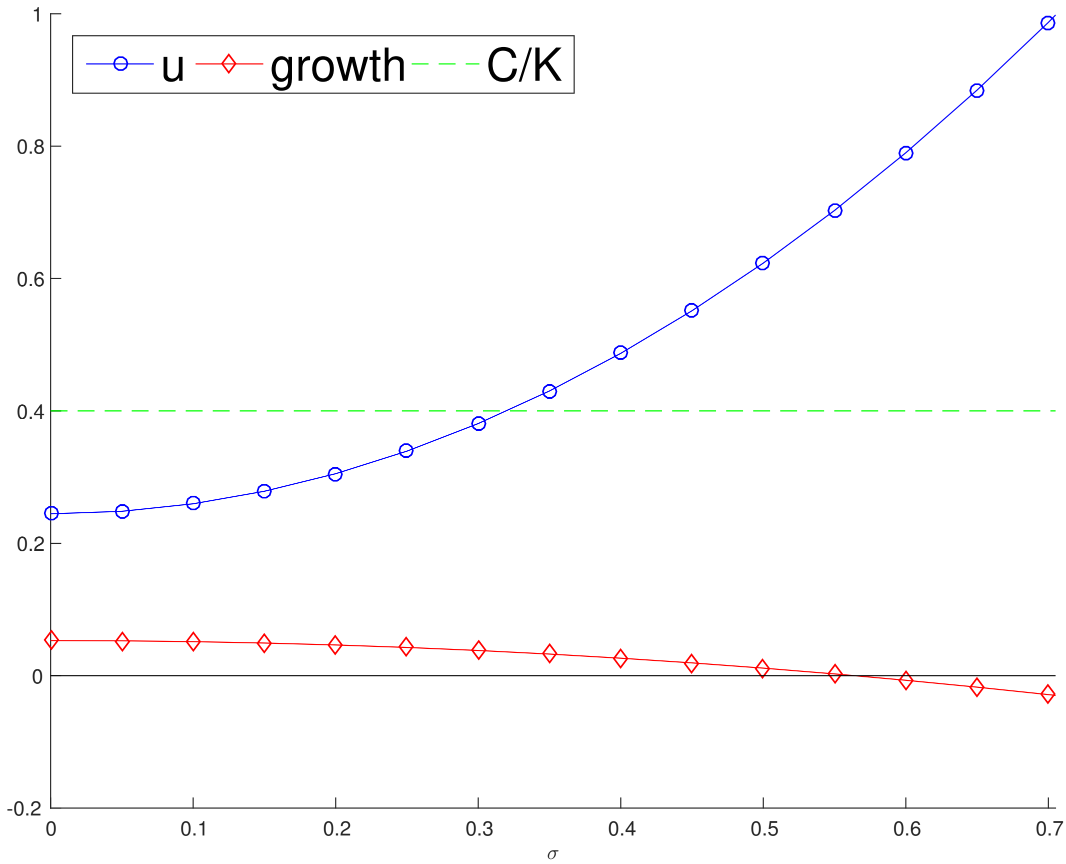

Figure 2.1 Human capital uncertainty and its effects on key variables.

Eq. (2.12) says that the time spent in working is constant as well, again consistent with Hiraguchi (2013). Note that it is increasing in σ_H. In other words, higher σ_H causes people to spend more of their time in working, and in parallel, less in the accumulation of human capital. Thus, higher human capital uncertainty leads to its contraction, consistent with Levhari and Weiss (1974) and Krebs (2003). This has an implication for the expected growth rate of human capital. Although stochasticity doesn't allow us to calculate the *actual* growth rate of human capital, we can nonetheless compute its *expected* growth rate $\mathcal{G}_H$, as u turns out to be constant. From Eq. (2.1), It is

$$\mathcal{G}_H \equiv \mathcal{E}\left(\frac{\dot{H}}{H}\right) = \frac{b - \rho - \delta_H - \frac{\sigma_H^2}{2}\alpha(1-\alpha) + \mu\gamma}{1-\alpha}, \tag{2.13}$$

where $\dot{H} \equiv dH/dt$. We can see that it is decreasing in human capital uncertainty

σ_H. As indicated above, higher σ_H discourages people to spend time in accumulating their human capital, and hence human capital contraction. It thus lowers expected growth rate of human capital. We can also see that, in the absence of technological progress ($\mu = \gamma = 0$), no depreciation ($\delta_H = 0$), and no uncertainty ($\sigma_H = 0$), the sign of growth rate is *solely* determined by the relative size of b and ρ (that is, $b \gtreqless \rho$). As Kuwahara (2017) discusses, it is the standard property of a deterministic Uzawa-Lucas model, and my model retains that property.

2.2.3.3 Numerical Example

It might be interesting to see whether $\mathcal{G}_H$ is positive under reasonable parameterization. It is positive if the numerator of (2.13) is positive; that is, if the inequality $\rho < b - \delta_H - \alpha(1-\alpha)\sigma_H^2/2 + \mu\gamma$ holds. It is illustrated by the line with diamonds in Fig. 2.1 (the line with circles indicates the relationship between u and σ_H). We see that, for the quite moderate degree of uncertainty (roughly $\sigma_H < 0.55$), $\mathcal{G}_H$ is positive. It gets, however, negative when uncertainty is larger than the threshold value. As such, when human capital uncertainty is sufficiently large, it may be difficult to realize the positive human capital growth. It can be overcome though, for instance, by raising the grow rate of technology μ via investment in R&D.

2.2.3.4 Welfare

Since we have the closed-form representation of a value function (2.9), our welfare analysis is possible by simply differentiating (2.9) with respect to the parameter of interest. This is one reason why an analytical solution is better than numerical solution: it allows us to reveal the welfare implications in the most transparent way. Specifically, first, we have

$$\frac{\partial J(K,H,A)}{\partial \mu} > 0, \tag{2.14}$$

in other words, technological progress is welfare-improving. This is missing in Krebs (2003). To grasp why technological progress improves welfare, note that the constant (2.10) is increasing in μ. Noting that A and H are multiplicative, technological progress strengthens the contribution of *both* A and H to the welfare J, as J is the positive function of state variables A and H.

Notice also that u is decreasing in μ in Eq. (2.12). This means that technological progress discourages people to spend time in working, or in parallel, encourages them to spend time in the accumulation of human capital. Since it increases the stock of human capital in the economy H, it improves welfare. Through these channels, technological progress generates a welfare gain.

Next, I examine the nexus between human capital uncertainty and welfare. We see

$$\frac{\partial J(K,H,A)}{\partial \sigma_H} < 0, \tag{2.15}$$

that is, higher human capital uncertainty deteriorates welfare, as in Krebs (2003). Why? First, because the constant (2.10) is decreasing in σ_H, higher uncertainty reduces both the contribution of A and H to welfare J (due to nonseparability). Second, since u is increasing in σ_H, higher human capital uncertainty encourages people to work more, or put differently, discourages them to spend time in the accumulation of human capital. This leads to human capital contraction, and thus welfare is deteriorated. Through these two channels, in contrast to technological progress, human capital uncertainty reduces welfare. This finding therefore complements Krebs (2003).

2.2.3.5 Simple Simulation

This thesis is concerned with the stochastic accumulation of human capital. Although studies cited above document the considerable degree of uncertainty associated with human capital, it would be still necessary to provide further empirical rationale for why I use a *stochastic* (not deterministic) differential equation (2.1). To motivate, see Fig. 2.2. It displays measures of U.S. human capital stock for people aged 15 to 64 in 1940s (in 5-year intervals), recently constructed by Lee and Lee (2016).

In general, human capital *accumulates* over time. Fig. 2.2, however, shows that in 1940s, there was human capital *contraction*. Although human capital had accumulated from 1940 to 1945, we can see its contraction from 1945 to 1950. At best, we can guess that it was caused by the large event such as World War II. The purpose of the exercise here is to see whether a stochastic differential equation

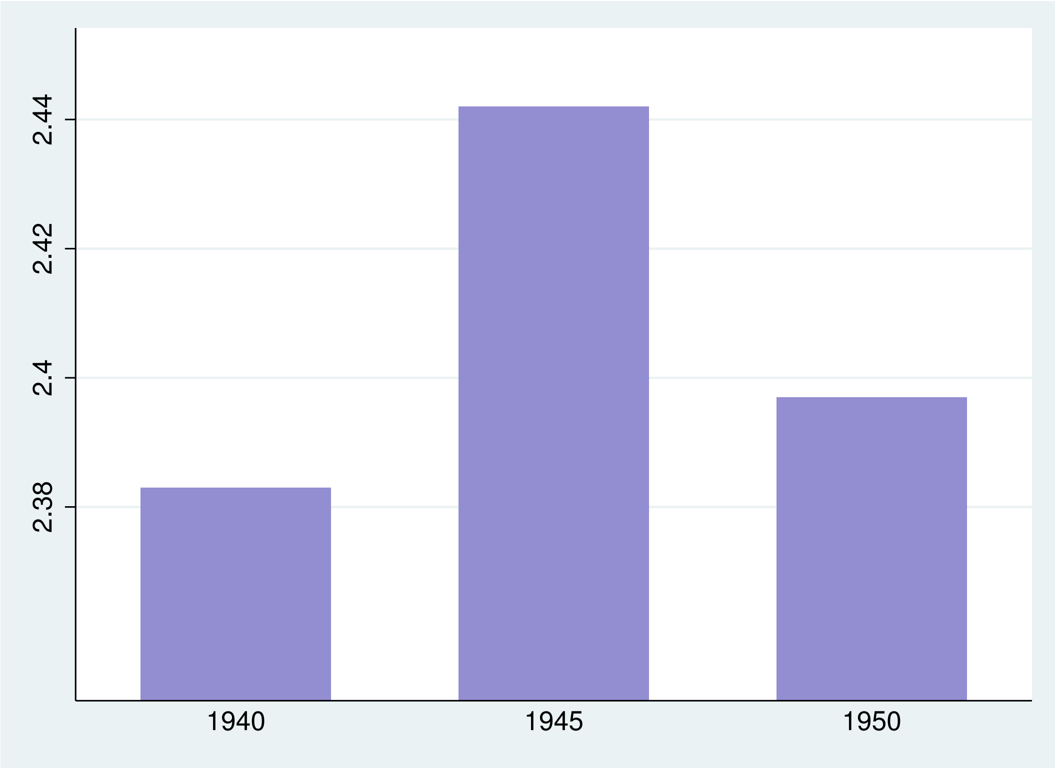

Figure 2.2 U.S. human capital stock in 1940s.

(2.1) can generate the bell-shaped pattern in Fig. 2.2. If a stochastic differential equation can better mimic Fig. 2.2 than its deterministic counterpart, it may justify why we need the stochastic elements in (2.1), in addition to the empirical studies such as Hartog et al. (2007) and Bilkic et al. (2012).

To this end, I present the discretized sample paths of Eq. (2.1) in Fig. 2.3. Specifically, I use the solution to Eq. (2.1):

$$H(t) = H(0)e^{\left(b(1-u)-\delta_H-\frac{\sigma_H^2}{2}\right)t}e^{\sigma_H z_H(t)},$$

for simulation.[1] It displays the simulated paths with various degrees of σ_H (1%, 10%, 20%, and 30%). First, see the line with $\sigma_H = 1\%$: it virtually represents a deterministic path. Over the simulated interval, it's always going up. For the reasonable parameter values, however, it doesn't replicate human capital contraction between 1945 and 1950. Thus, a deterministic differential equation *fails* to mimic Fig. 2.2.

[1]Higham (2001) provides a concise explanation of simulation technique for a stochastic differential equation driven by a Brownian motion process. See Appendix 2.C for how to solve a stochastic differential equation.

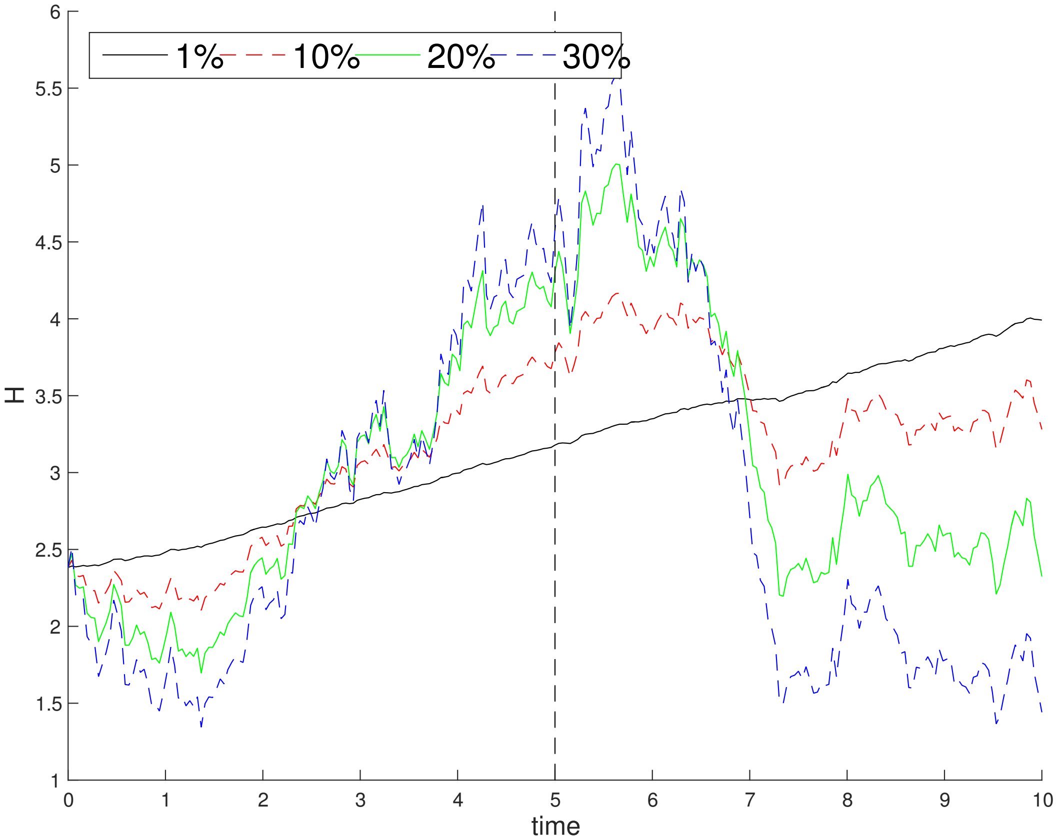

Figure 2.3 Simulation of a stochastic differential equation (2.1).

On the other hand, stochastic paths are successful in replicating bell-shaped curves. For $\sigma_H \geq 10\%$, we can first see the accumulation of human capital between $t = 0$ and $t = 5$ (albeit initial contraction at $t = 1$), and then the contraction between $t = 5$ and $t = 10$; hence generating a bell-shaped curve (especially for $\sigma_H = 20\%$ or 30%). The bottom line of this exercise is that, it might be appropriate to use a *stochastic* differential equation (2.1) rather than a deterministic differential equation to better account for the accumulation of human capital.

According to the Lee and Lee (2016) data on human capital stock, we can also see the "bell-shaped" pattern in the U.S. between 2000 and 2005. During this period, it declined from 3.71 (in 2000) to 3.67 (in 2005). In fact, this phenomenon isn't unique in the U.S. For instance, in Switzerland between 1980 and 2000, the number declined from 3.10 (in 1980) to 2.76 (in 2000); in Spain between 1915 and 1920, the number declined from 1.40 (in 1915) to 1.39 (in 1920); in Portugal between 2000 and 2005, the number declined from 2.32 (in 2000) to 2.25 (in 2005). These empirical evidence suggests that the exercise above can be applied not only to the period of unprecedentedly big events (such as World War II) or to a specific country, but also to other disruptive events across time and space.

The findings in this section can be summarized as follows:

Proposition 2.1. *I find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of human capital follows a Brownian motion process. Higher human capital uncertainty doesn't affect a consumption-capital ratio, increases time spent in working, reduces the expected growth rate of human capital, and deteriorates welfare.*

2.3 The Model with Jumps

In the previous section, we find the closed-form solution to the stochastic Uzawa-Lucas model with human capital accumulation following a Brownian motion process only. Despite the simulation exercise above, the accumulation of human capital may better be described by a *jump* process, rather than a Brownian motion process. Consequently, this section extends the model above. Specifically, as in Wälde (2011a) and Hiraguchi (2014), I consider the *mixture* of a Brownian motion process and many Poisson jump processes.

In the context of endogenous growth models, Poisson jump processes are frequently used, for instance, in the creative destruction or Schumpeterian growth model of Aghion and Howitt (1992). They are also theoretically studied by Sennewald and Wälde (2006) and Sennewald (2007) in detail. Existing studies on the stochastic Uzawa-Lucas model, such as Bucci et al (2011), Marsiglio and La Torre (2012a; b) and Hiraguchi (2013), however, all analyze a Brownian motion case *only*. In this section, I analyze whether we can still find the closed-form solution to the stochastic Uzawa-Lucas model with a combination of a Brownian motion process and many Poisson jump processes, and discuss the welfare implications of Poisson jump processes.[1]

2.3.1 Brownian Motion and Poisson Jump Process

Suppose that there are N independent Poisson jump processes $q_i(t)$ with the mean arrival rate λ_i that drive the accumulation of human capital, in addition to a Brownian motion process. The former occurs infrequently, while the latter goes on all the time. Then, a stochastic differential equation (2.1) is modified as a jump-diffusion process:

$$dH(t) = b(1-u(t))H(t)dt - \delta_H H(t)dt + \sigma_H H(t)dz_H(t) + \sum_{i=1}^{N} H(t)\beta_i dq_i(t), \quad (2.16)$$

where $\beta_i > -1$ is the size of jumps. During a time interval of infinitesimal length dt, the probability that a jump will occur is given by $\lambda_i dt$, and the probability that a jump will not occur is given by $1 - \lambda_i dt$; that is, $dq_i = \beta_i$ with probability $\lambda_i dt$, while $dq_i = 0$ with probability $1 - \lambda_i dt$.

The rest of the model remains unchanged. A representative household maximizes its expected utility (2.4) subject to the law of motion for physical capital

[1] Steger (2005) compares a Brownian motion process with a Poisson jump process in an AK model. He shows that a sensible comparison between these requires some unrealistic restrictions. Furthermore, even when they are imposed, he finds that insights from the comparison is *quantitatively negligible*. Following his findings, I won't do empirical simulation in what follows. In principle, with Poisson jump processes, we would see occasional jumps in Fig. 2.3, in addition to random fluctuations driven by a Brownian motion process.

(2.2) and technological progress (2.3), and to the stochastic process (2.16). Since first-order conditions (2.6) and (2.7) are unchanged, a resulting maximized HJB equation is given by

$$\begin{aligned}\rho J(K,H,A) &= \frac{\phi}{1-\phi}J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha}\right) A^{\frac{\gamma}{1-\alpha}} K^{\frac{\beta}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} \\ &- J_K\delta_K K + J_A\mu A + J_H bH - J_H\delta_H H + \frac{J_{HH}H^2\sigma_H^2}{2} \\ &+ \underbrace{\sum_{i=1}^{N}\lambda_i \left(J(K,(1+\beta_i)H,A) - J(K,H,A)\right)}_{\text{Poisson uncertainty}},\end{aligned}$$

where the last term is due to Poisson uncertainty. Because of a combination of a Brownian motion process and many Poisson jump processes, our analysis here becomes more complex; is it still possible to find the closed-form solution in this case? The results are as follows:

Theorem 2.2. *If we impose a parameter constraint (2.8), then there exists the closed-form representation of the value function that satisfies the TVC of the form*

$$J(K,H,A) = \mathcal{O}_X K^{\alpha+\gamma} + \mathcal{O}_Y H^{\alpha} A^{\gamma} + \mathcal{O}_Z,$$

where $\mathcal{O}_X = \mathbb{X}$, $\mathcal{O}_Z = \mathbb{Z}$, and

$$\begin{aligned}\mathcal{O}_Y &\equiv \frac{1}{b^{\alpha}}\left(\frac{\beta}{\rho+(\alpha+\gamma)\delta_K}\right)^{\beta} \\ &\times \left(\frac{1-\alpha}{\rho-\mu\gamma-\alpha b+\alpha\delta_H+\frac{\sigma_H^2}{2}\alpha(1-\alpha)-\sum_{i=1}^{N}\lambda_i\left((1+\beta_i)^{\alpha}-1\right)}\right)^{1-\alpha}.\end{aligned} \tag{2.17}$$

Moreover, the control variable C is still given by Eq. (2.11), while u is now expressed as

$$u = \frac{\rho-\mu\gamma-\alpha b+\alpha\delta_H+\frac{\sigma_H^2}{2}\alpha(1-\alpha)-\sum_{i=1}^{N}\lambda_i\left((1+\beta_i)^{\alpha}-1\right)}{b(1-\alpha)}. \tag{2.18}$$

Proof. See Appendix 2.A[1]

2.3.2 Macroeconomic Implications

Theorem 2.2 shows that we can still obtain a closed-form solution to the stochastic Uzawa-Lucas model even with a *combination* of a Brownian motion process and many Poisson jump processes. This finding crucially differs from previous studies such as Bucci et al (2011), Marsiglio and La Torre (2012a; b) and Hiraguchi (2013), because they consider a Brownian motion process only. As in the previous section, I in turn comment on main points in Theorem 2. To save space, I won't repeat what we discussed above.

In Eq. (2.18), we see that u is decreasing both in the arrival rate λ_i and jump size β_i. Therefore, increase in the arrival rate or jump size discourages people to spend time in working, and in parallel, encourages them to spend their time in a human capital sector. This leads to the accumulation of human capital. Because the human capital stock in an economy increases, the expected growth rate of human capital increases as well, and welfare is improved.

Formally, the expected growth rate of human capital with a Poisson jump $\mathcal{G}_H^q$ is now given by

$$\mathcal{G}_H^q \equiv \mathcal{E}\left(\frac{\dot{H}}{H}\right)$$
$$= \frac{b-\rho-\delta_H-\frac{\sigma_H^2}{2}\alpha(1-\alpha)+\mu\gamma+\sum_{i=1}^N \lambda_i((1+\beta_i)^\alpha-1)+(1-\alpha)\sum_{i=1}^N \lambda_i\beta_i}{1-\alpha},$$

where I use the "fact" that $\mathcal{E}(dq_i(t)) = \lambda_i dt$ (see Sennewald and Wälde, 2006). We can immediately see that $\mathcal{G}_H^q$ gets higher as the arrival rate λ_i and jump size β_i increase, because these lead to human capital accumulation. In the same vein,

[1]The condition for $u \in (0,1)$ is

$$\alpha(b-\delta_H)-\frac{\sigma_H^2}{2}\alpha(1-\alpha)+\mu\gamma+\sum_{i=1}^N \lambda_i\left((1+\beta_i)^\alpha-1\right) < \rho < b-\alpha\delta_H-\frac{\sigma_H^2}{2}\alpha(1-\alpha)+\mu\gamma+\sum_{i=1}^N \lambda_i\left((1+\beta_i)^\alpha-1\right). \tag{2.19}$$

Moreover, one can establish that the appropriate TVC is satisfied: Sennewald (2007) provides the proof of the TVC for a Poisson jump case.

it is straightforward to show that, for all i, we have

$$\frac{\partial J(K, H, A)}{\partial \lambda_i} > 0, \quad \frac{\partial J(K, H, A)}{\partial \beta_i} > 0.$$

So, a higher arrival rate and a larger jump improve welfare. As in the previous section, there are two underlying channels through which they improve welfare. First, via a control variable u, human capital accumulates. Second, via the constant (2.17), the contribution of both technology A and human capital H to welfare J are increased. It is worth reiterating that these results are possible only by considering a Poisson jump process. The findings of this section can be summarized as follows:

Proposition 2.2. *I find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of human capital follows a combination of a Brownian motion process and many Poisson jump processes. The higher arrival rate and larger size of jump decrease time spent in working, raise the expected growth rate of human capital, and improve welfare.*

Although this section extends the previous section, for example, by revealing the relationship between Poisson arrival rates and welfare, you may notice that one thing remains unresolved; as we saw in Fig. 2.1, a consumption-capital ratio C/K is still independent of human capital uncertainty, arrival rates, and a jump size.

2.4 The Model with Risky Physical Capital

How can we solve the puzzle that a consumption-capital ratio C/K is independent of major parameters of stochastic processes? The resolution would be obtained by realizing that, literally, consumption is the function of physical capital K. Therefore, if shocks had an impact on consumption, it would affect via K. For that reason, in this section, following Krebs (2003, Appendix 2), I assume that *both* human and physical capital accumulation follow the stochastic process. In Krebs (2003), the purpose of this extension is to check the robustness of his findings. My purpose here, however, is to solve a puzzle, and importantly, to find a closed-form solution to the stochastic Uzawa-Lucas model with risky physical and human capital accumulation.[1]

2.4.1 Risky Capital and Closed-Form Solution

Suppose that there are n independent Poisson jump processes q_j^k with arrival rates λ_j^k that drive the accumulation of *physical* capital. Then, a jump-diffusion process is

$$
\begin{aligned}
dK(t) = &\underbrace{(u(t)H(t))^{\alpha} K(t)^{\beta} A(t)^{\gamma}}_{\equiv Y(t)} dt - \delta_K K(t)dt - C(t)dt - \sigma_K K(t)dz_K(t) \\
&- \sum_{j=1}^{n} \beta_j^k K(t)dq_j^k(t),
\end{aligned} \tag{2.20}
$$

where, for simplicity, I assume that dz_H and dz_K are uncorrelated. $\sigma_K \geq 0$ is the diffusion coefficient of physical capital. $\beta_j^k \in (0, 1)$ is the jump size of the Poisson process for physical capital. Eq. (2.20) coincides with Eq. (12) in Wälde (2011a), if the production function $Y(t)$ is an AK type. As in Wälde (2011a), physical capital accumulation follows a combination of a Brownian motion process and many Poisson jump processes; it thus generalizes the model of Eaton (1981) and Rebelo and Xie (1999).

The rest of the model again remains unchanged. As such, a representative household maximizes its expected utility (2.4) subject to the law of motion for technological progress (2.3), and two stochastic processes (2.16) and (2.20). As first-order conditions (2.6) and (2.7) are not changed, a maximized HJB equation

[1] The notion of stochastic physical capital accumulation is first proposed by Eaton (1981). He assumes that the depreciation rate of physical capital follows a Brownian motion process. Similarly, Rebelo and Xie (1999) assume that it follows both a Brownian motion process and one Poisson jump process.

reads

$$\begin{aligned}\rho J(K,H,A) =& \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha}\right) A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} \\ & - J_K \delta_K K + J_A \mu A + J_H bH - J_H \delta_H H + \frac{J_{HH} H^2 \sigma_H^2}{2} + \frac{J_{KK} \sigma_K^2}{2} \\ & + \sum_{i=1}^{N} \lambda_i \left(J(K,(1+\beta_i)H,A) - J(K,H,A)\right) \\ & + \sum_{j=1}^{n} \lambda_j^k (J((1-\beta_j^k)K,H,A) - J(K,H,A)),\end{aligned} \tag{2.21}$$

where $J_{KK} \equiv \partial J^2/\partial K^2$ and the last two terms emerge out of uncertainty about depreciation of physical capital. Despite the complexity of Eq. (2.21), an explicit solution is available with one parameter restriction. It can be summarized as follows:

Theorem 2.3. *If we impose a parameter constraint (2.8), then there exists the closed-form representation of the value function that satisfies the TVC of the form*

$$J(K,H,A) = \mathcal{B}_X K^{\alpha+\gamma} + \mathcal{B}_Y H^\alpha A^\gamma + \mathcal{B}_Z,$$

where $\mathcal{B}_Z = \mathcal{Q}_Z$,

$$\mathcal{B}_X \equiv \frac{1}{\alpha+\gamma} \left(\frac{\beta}{\rho + (\alpha+\gamma)\delta_K + \frac{\sigma_K^2}{2}\beta(\alpha+\gamma) - \sum_{j=1}^{n} \lambda_j^k ((1-\beta_j^k)^{\alpha+\gamma} - 1)} \right)^{\beta}, \tag{2.22}$$

and

$$\mathcal{B}_Y \equiv \frac{(\alpha+\gamma)\mathcal{B}_X}{b^\alpha} \left(\frac{1-\alpha}{\rho - \mu\gamma - \alpha b + \alpha\delta_H + \frac{\sigma_H^2}{2}\alpha(1-\alpha) - \sum_{i=1}^{N} \lambda_i ((1+\beta_i)^\alpha - 1)} \right)^{1-\alpha}.$$

Besides, while u is still given by Eq. (2.18), the control variable C is expressed *as*

$$C = \frac{\rho + (\alpha+\gamma)\delta_K + \frac{\sigma_K^2 \beta(\alpha+\gamma)}{2} - \sum_{j=1}^{n} \lambda_j^k((1-\beta_j^k)^{\alpha+\gamma} - 1)}{\beta} K. \tag{2.23}$$

Proof. See Appendix 2.A[1]

2.4.2 Macroeconomic Implications

Theorem 3 demonstrates that we can still find the closed-form solution to the stochastic Uzawa-Lucas model *even in the presence of two kinds of stochastic processes that follow a combination of a Brownian motion process and many Poisson jump processes.* As in the previous sections, I won't repeat what we discussed above, and focus on the implications for consumption and welfare.

First, Eq. (2.23) shows that a consumption-capital ratio C/K is no longer independent of stochastic terms, as opposed to the horizontal dashed line in Fig. 2.1. Here, it *does* depend on physical capital uncertainty σ_K, its arrival rates λ_j^k and its jump size β_j^k (see also Propositions 6 and 7 in Rebelo and Xie (1999), and Eq. (16) in Wälde (2011a)). Taken together with Bucci et al (2011), Marsiglio and La Torre (2012a; b) and Hiraguchi (2013), it seems that the only shock type that can *directly* affect a consumption-capital ratio C/K in a stochastic Uzawa-Lucas model would be the stochastic process for physical capital. The reasonable guess then is that technology, demographic, and human capital shock have, in fact, nothing to do with the optimal ratio of consumption to physical capital.

Second, since welfare is the function of the state variable K − which is now stochastic − we can investigate the relationship between welfare and uncertainty about physical capital accumulation. Unlike Krebs (2003), because households do not have choice about investment in physical capital, the channel through which shocks affect welfare can be identified via the constant (2.22).[2] One can show that

$$\frac{\partial J(K,H,A)}{\partial \sigma_K} < 0,$$

[1] The condition for $u \in (0,1)$ is still given by the inequality (2.19).

[2] To be precise, as $\mathcal{B}_X$ is contained in $\mathcal{B}_Y$, shock terms in (2.22) affect both A and H, and hence welfare J. This channel, however, would be too obvious to explain in detail.

and that, for all j,

$$\frac{\partial J(K,H,A)}{\partial \lambda_j^k} < 0, \quad \frac{\partial J(K,H,A)}{\partial \beta_j^k} < 0.$$

So, the larger physical capital shock, higher arrival rates, and a larger size of jump all reduce welfare, because it leads to physical capital contraction. It is worth emphasizing that no studies have found a closed-form solution to the stochastic Uzawa-Lucas model with depreciation of physical capital following a Brownian motion process and many Poisson jump processes. The findings in this section can be summarized as follows:

Proposition 2.3. *I find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of both human and physical capital follows a combination of a Brownian motion process and many Poisson jump processes. The larger physical capital shock, higher arrival rates, and a larger size of jump deteriorate welfare. More importantly, consumption-capital ratio depends on shock terms when the depreciation of physical capital is driven by a stochastic process.*

2.5 Concluding Remarks

This chapter has solved and explained the baseline model of this thesis. As we saw, higher human capital uncertainty lowers economic growth. Thus, one of four empirical patterns in Ch. 1 – a *negative* link – has been theoretically proved. We will study the rest of three patterns in the following chapters.

The strong assumption in this chapter is the *absence* of stochastic processes, though this is in part to facilitate an exposition. By relaxing this assumption, we will see that positive or nonlinear associations can be replicated, depending on how stochastic processes are correlated. In subsequent chapters, we will abstract from Poisson jump processes to make the mechanism transparent.

2.A Value Function

This appendix briefly describes how to find the closed-form representation of the value function in Theorems 1, 2, and 3. For this purpose, postulate the tentative value function of the form

$$J(K,H,A) = \mathbb{X}K^{\theta_1} + \mathbb{Y}H^{\theta_2}A^{\theta_3} + \mathbb{Z},$$

where $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, θ_1, θ_2, and θ_3 are all unknown constants to be determined. The relevant partials are $J_K = \mathbb{X}\theta_1 K^{\theta_1 - 1}$, $J_{KK} = \mathbb{X}\theta_1(\theta_1 - 1)K^{\theta_1 - 2}$, $J_A = \mathbb{Y}\theta_3 H^{\theta_2} A^{\theta_3 - 1}$, $J_H = \mathbb{Y}\theta_2 H^{\theta_2 - 1} A^{\theta_3}$, and $J_{HH} = \mathbb{Y}\theta_2(\theta_2 - 1)H^{\theta_2 - 2}A^{\theta_3}$.

To obtain the explicit expression, substitute these partials into a maximized HJB equation (2.21). Then, set $\theta_1 = \alpha + \gamma$, $\theta_2 = \alpha$, and $\theta_3 = \gamma$. Finally, by imposing a parameter restriction (2.8), you can find the explicit expressions for $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$, and consequently, those for control variables C and u and the value function $J(K,H,A)$ in Theorem 3. Expressions in Theorem 2 are available by abstracting from the shock terms associated with the stochastic depreciation of physical capital, while those in Theorem 1 are obtained by abstracting from many Poisson jump processes in Eq. (2.16).

2.B Stochastic Transversality Condition

In this Appendix, I prove a stochastic transversality condition (TVC):[1]

$$\lim_{t\to\infty} \mathbb{E}[e^{-\rho t}K^{1-\phi}] = \lim_{t\to\infty} \mathbb{E}[e^{-\rho t}H^{\alpha}A^{\gamma}] = 0.$$

I first show $\lim_{t\to\infty}\mathbb{E}[e^{-\rho t}H^{\alpha}A^{\gamma}] = 0$. Using the solution to Eq. (2.1) and technique in Bucci et al. (2011, footnote 6), we have

$$\mathbb{E}[H(t)^{\alpha}] = e^{\alpha\left(b(1-u)-\delta_H - \frac{\sigma_H^2}{2}(1-\alpha)\right)t}.$$

Using this, calculate the growth rate of the product $e^{-\rho t}H^{\alpha}A^{\gamma}$:

[1]See Chang (2004, Ch.4) for related mathematics. The proof here is based on Appendix A of Bucci et al. (2011) and Appendix B of Hiraguchi (2013). In their model, however, it's technological progress that follows a Brownian motion process.

$$\begin{aligned}
&-\rho+\alpha\left(b(1-u)-\delta_H-\frac{\sigma_H^2}{2}(1-\alpha)\right)+\gamma\mu \\
&=-\rho+\alpha b-\frac{\alpha}{1-\alpha}\left(\rho-\gamma\mu-\alpha b+\alpha\delta_H+\frac{\sigma_H^2}{2}\alpha(1-\alpha)\right)-\alpha\delta_H \\
&-\frac{\sigma_H^2}{2}\alpha(1-\alpha)+\gamma\mu.
\end{aligned}$$

To satisfy TVC, this must be negative. So,

$$-\rho+\alpha b-\frac{\alpha}{1-\alpha}\left(\rho-\gamma\mu-\alpha b+\alpha\delta_H+\frac{\sigma_H^2}{2}\alpha(1-\alpha)\right)-\alpha\delta_H-\frac{\sigma_H^2}{2}\alpha(1-\alpha)+\gamma\mu<0.$$

Rewriting this, we find

$$\rho>\alpha(b-\delta_H)+\mu\gamma-\alpha(1-\alpha)\frac{\sigma_H^2}{2}.$$

Indeed, this is the condition for $u>0$ in Theorem 1. Thus, we have established $\lim_{t\to\infty}\mathbb{E}[e^{-\rho t}H^\alpha A^\gamma]=0$. Next, I show $\lim_{t\to\infty}\mathbb{E}[e^{-\rho t}K^{1-\phi}]=0$. For this purpose, let $\mathcal{K}\equiv e^{-\rho t}K^{1-\phi}$. From Eqs. (2.2), (2.11), and (2.12), the growth rate of $\mathcal{K}$ is

$$\begin{aligned}
\frac{\dot{\mathcal{K}}}{\mathcal{K}}&=-\rho+(1-\phi)\frac{\dot K}{K} \\
&=-\rho+(1-\phi)\left((uH)^\alpha K^{\phi-1}A^\gamma-\delta_K-\frac{\rho+(1-\beta)\delta_K}{\beta}\right) \\
&=-\rho+(1-\phi)\left(\frac{(uH)^\alpha A^\gamma e^{-\rho t}}{\mathcal{K}}-\frac{\rho+\delta_K}{\beta}\right) \\
&=(1-\phi)\left(\frac{(uH)^\alpha A^\gamma e^{-\rho t}}{\mathcal{K}}\right)-\frac{\rho+(1-\phi)\delta_K}{\phi},
\end{aligned}$$

where $\dot{\mathcal{K}}\equiv d\mathcal{K}/dt$. Therefore, we have

$$\begin{aligned}
\dot{\mathcal{K}}&=(1-\phi)(uH)^\alpha A^\gamma e^{-\rho t}-\underbrace{\rho+(1-\phi)\delta_K}_{\equiv\Omega}\mathcal{K} \\
&=-\Omega\mathcal{K}+(1-\phi)e^{-\rho t}u^\alpha A(0)^\gamma e^{\mu\gamma t}H^\alpha.
\end{aligned}$$

Solving this differential equation forwards and taking expectations, we finally get

$$\mathbb{E}[K(t)] = e^{-\Omega t}\left(\mathcal{K}(0) + (1-\phi)u^{\alpha}A(0)^{\gamma}\int_0^t e^{-(\rho-\mu\gamma-\Omega)s}\mathbb{E}[H(t)^{\alpha}]ds\right).$$

Because $\Omega > 0$, we have established $\lim_{t\to\infty}\mathbb{E}[\mathcal{K}] = 0$ or $\lim_{t\to\infty}\mathbb{E}[e^{-\rho t}K^{1-\phi}] = 0$. As a result, a value function satisfies TVC.

2.C Solution to the Stochastic Differential Equation

In this Appendix, I show that a solution to the stochastic differential equation of the form

$$dL(t) = nL(t)dt + \sigma L(t)dz(t),$$

is given by

$$L(t) = L(0)e^{\left(n-\frac{\sigma^2}{2}\right)}e^{\sigma z(t)}. \tag{2.24}$$

Let $y(t) \equiv logL(t)$. Using Itô's lemma,

$$\begin{aligned} dy(t) &= \frac{\partial y(t)}{\partial L(t)}dL(t) + \frac{1}{2}\frac{\partial^2 y(t)}{\partial L(t)^2}(dL(t))^2 = \frac{dL(t)}{L(t)} - \frac{1}{2}\frac{1}{(L(t))^2}(dL(t))^2 \\ &= (ndt + \sigma dz(t)) - \frac{1}{2(L(t))^2}\sigma^2(L(t))^2dt = ndt + \sigma dz(t) - \frac{\sigma^2}{2}dt \\ &= \left(n - \frac{\sigma^2}{2}\right)dt + \sigma dz(t). \end{aligned}$$

As

$$\int_0^t d(logL(s)) = logL(t) - logL(0),$$

we have

$$logL(t) - logL(0) = \int_0^t \left(n - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dz(s).$$

Noting that $z(0) = 0$, we get

$$log\left(\frac{L(t)}{L(0)}\right) = \left(n - \frac{\sigma^2}{2}\right) t + \sigma z(t).$$

Using the rule $e^{logX} = X$, we finally have

$$\frac{L(t)}{L(0)} = e^{\left(n - \frac{\sigma^2}{2}\right)t + \sigma z(t)}.$$

Rearranging this, we get Eq. (2.24). Therefore, if $L(0) > 0$ (as it must in economics), $L(t) > 0$ for all t with probability 1. Put differently, $L(t)$ is bounded from below, even though a Brownian motion $z(t)$ is unbounded.

At the same time, if we assume σ is independent of $L(t)$, that is, if

$$dL(t) = nL(t)dt + \sigma dz(t),$$

then, its solution is

$$L(t) = L(0) + n\int_0^t L(s)ds + \sigma z(t).$$

As a result, an assumption of $L(0) > 0$ doesn't exclude a possibility of negative $L(t)$ when $z(t) \to -\infty$. See Chang (2004) for a more in-depth treatment.

2.D Closed-form Solution to Stochastic Growth Models: Literature

This Appendix surveys some studies that have tried finding the closed-form solution to a stochastic growth model.[1] Stochastic growth models are often intractable and involve a certain amount of technical apparatus not familiar to economists (Turnovsky, 2000, p.580). Therefore, as Smith (2007, p.1) puts, "Paradoxically, the rise of computational methods has put a premium on analytical solutions..."

[1] See Introduction of Bucci et al. (2011) and Wälde (2011) for more on this subject and related literature.

and some studies have attempted to know which stochastic growth model possibly admits a closed-form solution, and if so, under what condition(s).

Rebelo and Xie (1999) study a stochastic, monetary growth model with AK technology in which (physical) capital accumulation follows a combination of a Brownian motion and a Poisson jump process. They then analytically solve their model by using the Xie (1991; 1994) condition; that is, by confining the risk aversion parameter to physical capital share of income. Smith (2007) finds a closed-form solution to a one-sector, stochastic Ramsey growth model. Imposing the same condition, Bucci et al. (2011) and Hiraguchi (2013) analytically solve the stochastic, two-sector endogenous growth model of Uzawa (1965) and Lucas (1988) in which technological progress is driven by a geometric Brownian motion (GBM) process. By the same token, Marsiglio and La Torre (2012) obtain an explicit solution to the stochastic Uzawa-Lucas model in which population dynamics follows a GBM process. Posch and Wälde (2011) solve a "vintage capital" growth model featuring distortionary taxes under uncertainty. Hiraguchi (2014) finds a closed-form solution to a Ramsey model with leisure. Finally, Menoncin and Nembrini (2018) solve a stochastic growth model with hyperbolic absolute risk aversion (HARA) preferences.

None of studies cited here (except for Menoncin and Nembrini, 2018), however, takes a *correlation* of stochastic processes into account. Thus, they are unable to replicate a *nonlinear* relationship between growth and uncertainty. This is a job of subsequent chapters.

Chapter 3

Resource Scarcity, Technological Progress, and Stochastic Growth

Countries with more natural resources are likely to achieve higher growth and improve the welfare of people, as they can use natural resources such as coal, petroleum, and natural gas, to produce output, in addition to physical and human capital.[1] In parallel, we have to be conscious that most resources are *exhaustible*; they don't necessarily renew themselves at a sufficient rate. Unduly immoderate use of resources is impossible, at least from the long-run viewpoint, as the amount of natural resources on earth is fixed.

It is simple to imagine that, at some point in the future, we may use up all exhaustible resources on this globe. Unable to use resources in production, economic growth will slow down, and eventually we would have zero growth. With no growth afterward, economic welfare would deteriorate further and further. This is a worst-case scenario. If so, will growth slow down due to the constraint posed by resource scarcity, as predicted by AK models with natural resources (Aghion and Howitt, 2009, Ch.16)?

Despite this concern, this topic is absent in the masterly survey of Acemoglu (2009).[2] This may reflect the view that the depletion of resources is not urgent concern, if not negligible. Indeed, as Fig. 3.1 shows, the prices of natural resources (excluding energy and precious metals) have had a *declining* trend over the last 160

[1] This chapter is based on Tsuboi (2019b).

[2] Solow (2009) also points out the lack of studies on the growth-resource linkage in the growth literature.

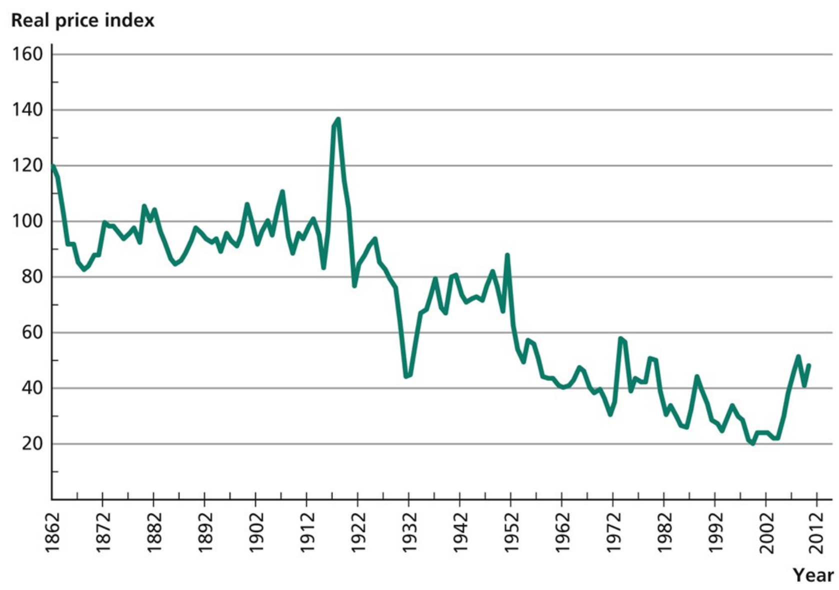

Figure 3.1 The prices of natural resources over the period 1850-2010.

years.[1] Note that, if we are really running out of natural resources, their prices must go *up*, rather than go *down*. Therefore, this *decline* in natural resource prices over the long run seems to tell us that the world is *not* running out of natural resources; indeed, Aguilera and Ripple (2012) estimate that, in Europe, oil and gas are more abundant than commonly thought.

There are at least *two* reasons why the scarcity of natural resources has been "postponed." First, Dasgupta and Heal (1974) argue that technological progress − the discovery of new substitutes − has made previously essential exhaustible resources *in*essential. On that account, as long as the discovery repeats, the depletion of resources won't pose a destructive problem. Second, as Weil (2013) puts, countries can make up for any resources they lack by simply importing them from abroad. Truly, if a resource-poor country needs petroleum in production, it can import petroleum from other countries. Due to the decline of transport costs over the last decades, the cost of replenishment via trade might not too great. This may allow countries to avoid problems that would arise out of resource scarcity.

But these two views are not strong enough to claim that *finite* nature of natural resources can be avoided for an *indefinite* period of time. On the first view,

[1] Prices are measured excluding energy and precious metals. The data are for a basket of commodities that has been altered over time to reflect changing demand in the most developed countries. Figs. 3.1 and 3.2 are from Weil (2013).

technological progress that substitutes today's essential resources may not happen, and it is the reason why Dasgupta and Heal (1974) use the *stochastic* model in which the arrival date of the discovery is *uncertain*. In the absence of the discovery of substitutes, their logic doesn't hold. On the second view, it is valid at the *country* level, but not at the *global* level. Unlike a single country, the world cannot make up for a shortage of natural resources by importing.

Fig. 3.2 shows the price of crude oil for the period 1861-2010. At first glance, we see that, as in Fig. 3.1, the price of oil has been low over the long run, especially between 1880 and 1970. In addition to some spikes due to political factors, such as the Iranian Revolution in 1979, however, we see that the price has basically gone *up* since 2000. Although the price of natural resources is hard to predict, there seems to be a good reason to worry about the recent surge in oil prices; it might be the signal of resource depletion.[1]

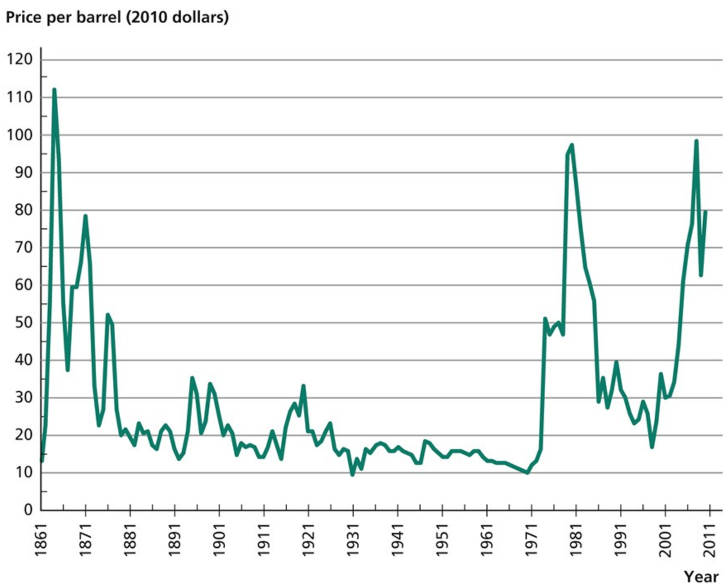

Figure 3.2 The price of crude oil over the period 1861-2010.

Some studies have explored the implications of resource scarcity for economic growth. Solow (1978) analyzes this linkage using the CES (constant elasticity of substitution) production function.[2] Focusing on the elasticity of substitution between resources and other inputs (physical capital and labor), he concludes that

[1]So far, we have examined the data on *prices*. Data for *quantities* are in Appendix 3.B.

[2]See also Solow (1974) for more on technical aspects of the association between growth and resources.

resource scarcity would not pose a serious problem from the empirical point of view. Cheviakov and Hartwick (2009) augment the Solow (1956) model by incorporating exhaustible resources. They show that the higher rate of depreciation of physical capital destroys an economy, whereas it can be avoided by strong technological progress. Vita (2007) extends the human-capital based endogenous growth model of Lucas (1988) by considering the substitutability between exhaustible resources and *secondary materials* – the manufactured material that has already been used at least once, and may be used again after recycling. He argues that varying substitutability affects economic growth rate during the transition path. Aghion and Howitt (2009, Ch.16) show that, even in the presence of exhaustible resources, growth can be sustained in the creative destruction (or Schumpeterian) growth model. Romer (2012, Ch.1) describes why technological progress would make it possible to sustain growth, even with the resource depletion and land in neoclassical growth models.

None of the studies above, however, has explored *uncertainty* in the dynamics of resources. It is recognized that resource dynamics is in part stochastic.[1] In a partial equilibirum model, Pindyck (1984) examines the impact of resource uncertainty (modelled as stochastic processes). Interestingly, he finds that effects of larger resource fluctuation on the extraction rate are *ambiguous*; higher resource uncertainty has the positive, zero, or negative influence, depending on the specification of the function governing the stochastic resource dynamics. Despite profound insights that can be gained by taking resource uncertainty associated into account, it is absent in recent growth-resource papers such as Vita (2007) and Cheviakov and Hartwick (2009).

To complement the studies cited above, I extend the *deterministic* Uzawa-Lucas growth model with exhaustible resources, developed by Vita (2007) and Neustroev (2014) independently.[2] Specifically, I present the *stochastic* Uzawa-Lucas model in which *both* technological progress and resource dynamics are driven by the correlated stochastic process, in the spirit of stochastic technological progress by Dasgupta and Heal (1974) and stochastic resource dynamics

[1] See Clark (1979), Pindyck (1980; 1984), and a number of references cited therein.

[2] For *renewable* resources, Nakamoto and Futagami (2016) present the dynamic, small open-economy growth model under certainty.

by Pindyck (1980; 1984). Besides, I consider the minimal degree of openness, so that there would be no room to import resources from abroad. This assumption allows me to derive the analytical solution even in the presence of two *correlated* stochastic processes. As all findings are characterized *in closed form*, the model's mechanism must be transparent.

More concretely, I analyze the stochastic two-sector endogenous growth model of Uzawa (1965) and Lucas (1988) in which the engine of growth is the accumulation of human capital. Uncertainty is modeled as a correlated Brownian motion process. As such, technological progress and resource dynamics are driven by stochastic processes. I then use those to examine how higher uncertainty affects economic growth and the welfare of agents. Intuitively, higher uncertainty reduces economic growth (see Ramey and Ramey, 1995) and deteriorates economic welfare. Indeed, in the baseline scenario, I find that higher uncertainty weakens economic growth and that welfare deteriorates. In contrast, when two stochastic processes of technological progress and resource dynamics are *positively* correlated, I show that there exists a *hump-shaped* relationship. This seems counterintuitive, but not so if one considers as follows: suppose that uncertainty gets higher. Then, firms would refrain from hiring and innovating, resulting in economic stagnation. As discussed in Bloom (2014), however, an equally possible scenario is that, faced with a more uncertain future, some firms appear *more* willing to innovate; that is, higher uncertainty can *stimulate* R&D. To innovate, as in the seminal paper of Romer (1990), firms need human capital − "skilled labor" or researchers.

In response to higher demand for human capital, households begin to spend more of their time in learning in a human capital sector, instead of working in a final-goods sector to produce. This encourages the further accumulation of human capital in the economy, enabling firms to employ human capital for R&D. This channel, as such, promotes technological progress, thereby raising the growth rate and improving welfare, as long as uncertainty is moderate. At the same time, when uncertainty gets much higher, the standard negative effects due to risk aversion comes into play and eventually dominates the former positive effect. Because of this *tensions* between two conflicting forces, the net result is indeterminate, yielding a hump-shaped pattern.

Summing up, the purpose of this chapter is to examine the implications of the natural resource scarcity and associated uncertainty for growth and welfare, by analytically solving the open, stochastic Uzawa-Lucas model in which both technological progress and resource dynamics are driven by stochastic processes. This chapter is organized as follows. Sect. 2 sets up the model and discusses its implications. Concluding remarks appear in Sect. 3.

3.1 The Model

In this section, I develop the stochastic Uzawa-Lucas model in which both technology and the depletion of exhaustible resources follow stochastic processes. Suppose that the world economy consists of N countries, indexed by $i = 1, ..., N$. Throughout, I assume that N is large enough so that each country is small relative to the rest of the world; it ignores its effect on world aggregates. Suppose also that the total number of workers in country i, L_i, equals unity in all countries so that per capita terms are equivalent aggregate terms in all countries. Throughout the paper, I often simplify the notation by suppressing time and country indices when this causes no confusion.

The latter assumption of $L_i = 1$ is also made in the closed economy of Bucci et al. (2011) and Hiraguchi (2013), as it greatly simplifies the analysis, and as population growth is not substance of their paper and this chapter. Besides, I suppose that each country admits a representative household. It is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u(t) \in (0, 1)$ denote the fraction of time spent working to produce final goods $Y(t)$. So, $1 - u(t)$ represents the fraction of time spent learning. The amount of leisure is fixed exogenously, so there is no choice about it.

3.1.1 Capital Accumulation and Resource

The law of motion for the accumulation of human capital accumulation in country i, $H_i(t)$, is given by

$$dH_i(t) = b(1 - u(t))H_i(t)dt - \delta_H H_i(t)dt, \tag{3.1}$$

where $b > 0$ is an exogenous parameter that indicates how efficient human capital accumulation is. $\delta_H \in (0,1)$ is its depreciation rate. Less $u(t)$ mirrors more $1-u(t)$, thereby accelerating the growth of human capital. I assume that the initial stock of human capital $H(0) = H_0 > 0$ is given.

Next, the resource constraint in country i takes the form

$$dK_i(t) = \underbrace{(u(t)H_i(t))^{\gamma} K_i(t)^{\beta} A_i(t)^{\alpha} (\vartheta_i \bar{S}(t))^{1-\alpha-\beta-\gamma}}_{\equiv Y_i(t)} dt - C_i(t)dt - \delta_K K_i(t)dt, \quad (3.2)$$

where $K_i(t)$ is physical capital. $\gamma \in (0,1)$ denotes the human capital share of income, $\beta \in (0,1)$ the physical capital share of income, and $\alpha \in (0,1)$ in the generalized Cobb-Douglas production function à la Mankiw et al. (1992). $\delta_K \in (0,1)$ is the depreciation rate of physical capital. $C_i(t)$ denotes consumption of the final good $Y_i(t)$. The initial stock of physical capital $K(0) = K_0 > 0$ is given as well.

$A_i(t)$ is technology in country i. As in Bucci et al. (2011) and Hiraguchi (2013), it follows a Brownian motion process[1]:

$$dA_i(t) = \mu A_i(t)dt + \sigma_a A_i(t)dz_a(t), \quad (3.3)$$

where μ denotes an exogenous growth rate of technology. $\sigma_a \geq 0$ is the diffusion coefficient of technology (if $\sigma_a = 0$, then we would recover the deterministic limit). dz_a is the increment of a Brownian motion process such that the mean $E(dz_a) = 0$ and variance $\mathcal{V}(dz_a) = dt$. As changes in the process over any finite interval of time are normally distributed, a variance increases linearly with the time interval dt. I assume that the initial stock of technology $A(0) = A_0 > 0$ is also given, so that $A_i(t) > 0$ for all t with probability 1.[2] These uncertainty explicitly capture the *random* arrival nature of technological progress, and thus develop an original idea of Dasgupta and Heal (1974) in a rigorous manner.

$S_i(t)$ denotes the amount of exhaustible resources available in country i at time

[1] See Tsuboi (2019b) for a more general version with the mixture of a Brownian motion process and many Poisson jump processes.

[2] See Dixit and Pindyck (1994, Ch.3) and Chang (2004) for a lucid account of a Brownian motion (or Wiener) process.

t, and $\bar{S}$ is the amount of the *global* stock of exhaustible resources (the bar indicates that a variable is measured on a global scale). ϑ_i is an exogenous parameter which denotes the share of resources country i can use in the production of final goods.

In the absence of uncertainty, following the closed economy model of Scholz and Ziemes (1999), the finite resource stock S at time τ would have been described by

$$S_i(\tau) = \int_{\tau}^{\infty} R_i(t)dt,$$

where $R_i(t)$ is all future extraction of the resource stock (or, differentiating with respect to time τ, we get $dS_i(t) = -R_i(t)dt$). In this chapter, there are N countries. So, the above expression should be modified as

$$\bar{S}(\tau) = \sum_{i=1}^{N} \int_{\tau}^{\infty} R_i(t)dt,$$

where now $R_i(t) \equiv \theta_i \bar{S}(t)$. To analyze the impacts of resource uncertainty, following Pindyck (1980; 1984), the law of motion for the global stock of exhaustible resources is also stochastically governed by a Brownian motion process:

$$d\bar{S}(t) = -\underbrace{R_i(t)}_{\equiv \theta_i \bar{S}(t)} dt + \sigma_s \bar{S}(t) dz_s(t), \tag{3.4}$$

where $\sigma_s \geq 0$ is the diffusion coefficient of exhaustible resources. dz_s is, again, the increment of a Brownian motion process such that the mean $E(dz_s) = 0$ and variance $\mathcal{V}(dz_s) = dt$. I assume that the initial stock of exhaustible resources $\bar{S}(0) = \bar{S}_0 > 0$ is given as well, so that $\bar{S}(t) > 0$ for all t with probability 1.

Unlike previous studies, the key assumption I make here is that two diffusion processes are *correlated*, that is, $(dz_a)(dz_s) = \eta dt$, with η being the correlation coefficient of dz_a and dz_s. We will see that η will play a vital role in anatomizing the implications of natural resource scarcity. Note that, technically, if $\eta = 0$ and $\sigma_s = 0$, then my model recovers that of Hiraguchi (2013). Before going on, as η is the most important parameter in this chapter, we need to understand what we should think of, say, the $\eta > 0$ case.

First, we can think of $\eta > 0$ as the case of *resource-saving* technological progress. This case seems realistic (and generates the most important finding), as history is full of examples of new technologies that have eased resource constraints that were impeding economic growth. For example, Weil (2013, Ch.16) provides four case studies of resource-saving technologies: nitrogen, rubber, nuclear fusion, and solar energy. In general, higher uncertainty appears to reduce the willingness of firms to hire and invest. As Bloom (2014) discusses, however, some empirical studies find the opposite case; higher uncertainty can *stimulate* R&D because some firms appear *more* willing to innovate in the face of a more uncertain future. If the incentive of firms to innovate gets stronger in response to higher uncertainty under the $\eta > 0$ scenario, resource scarcity may be "postponed" thanks to the invention of the resource-saving technologies. In sum, the $\eta > 0$ scenario is the case where economic agents can possibly be "optimistic" as the resource-saving innovation may be promoted by firms in anticipation of a more uncertain future.

Second, we may think of $\eta < 0$ as the case of *resource-eating* technological progress. This case, in contrast, seems unrealistic (and will generate insights already discussed in the literature). In this scenario, even if firms possibly innovate in response to higher uncertainty, it will not undo (and rather worsen) the resource scarcity and economic agents will be "pessimistic" about the future as there is almost no prospect of overcoming the scarcity of resources. Overall, although η can be either positive or negative, it seems that we wish to center our discussion mainly on the $\eta > 0$ scenario instead of the $\eta < 0$ scenario when we undertake the growth and welfare analysis.

3.1.2 Household

By assuming the world economy consisting of a large number of N countries, there is no longer room for importing from abroad, hence dealing with the argument of Weil (2013). As ϑ_i is exogenous, I assume that there exists a world planner who decides how much of $\bar{S}$ country i can use at time t. This is clearly a strong assumption; it's more desirable to endogenize ϑ_i to model a strategic interaction among countries, such as the tragedy of the commons. Given the presence of *two, correlated* stochastic processes, however, the model is already too intractable to be solved in closed form. Therefore, to keep the model as simple and tractable as possible, I leave the endogenization of ϑ_i for future research.

Preferences of a representative household in country i at time $t = 0$ are given by the standard constant relative risk aversion (CRRA) utility:

$$\mathbb{E}\int_0^{\infty} e^{-\rho t}\frac{C_i(t)^{1-\phi}-1}{1-\phi}dt, \tag{3.5}$$

where $\mathbb{E}$ is the mathematical expectation operator with respect to the information set available to a representative household. $\rho > 0$ is its subjective discount rate; that is, the rate at which utility is discounted. $\phi > 0$ is the index of relative risk aversion (and $1/\phi$ is intertemporal elasticity of substitution). When future consumption is uncertain, a larger ϕ makes future utility gain smaller, raising the value of additional future consumption.

Summing up, a representative household in country i maximizes its expected utility (3.5) subject to the law of motion for the accumulation of human capital (3.1) and for physical capital (3.2), and to two stochastic processes for technological progress (3.3) and for global exhaustible resources (3.4).

3.1.3 Optimization

To solve this problem, let $J(K, A, H, \bar{S})$ denote a value function. Then, a corresponding Hamilton-Jacobi-Bellman (HJB) equation is

$$\begin{aligned}\rho J = {} & \frac{C_i(t)^{1-\phi}}{1-\phi} + \frac{\mathbb{E}}{dt} \\ & \times \left(J_K dK + J_A dA + J_H dH + J_S d\bar{S} + J_{AS}(dA)(d\bar{S}) + \frac{J_{AA}(dA)^2 + J_{SS}(d\bar{S})^2}{2} \right),\end{aligned}$$

where $J_X = \partial J/\partial X$, $J_{XX} = \partial^2 J/\partial X^2$, and $J_{XY} = \partial J/\partial X \partial Y$ for variables X and Y. As control variables are C_i and u, first-order conditions are

$$C = J_K^{-\frac{1}{\phi}}, \tag{3.6}$$

and

$$u = \frac{1}{H}\left(\frac{\gamma J_K A^{\alpha} K^{\beta}(\vartheta_i \bar{S})^{1-\alpha-\beta-\gamma}}{bJ_H}\right)^{\frac{1}{1-\gamma}}. \tag{3.7}$$

Substituting these first-order conditions (3.6) and (3.7) into the above HJB equation, after some algebra, we arrive at

$$\begin{aligned}
0 =& \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - J_K \delta_K K - \frac{1}{1-\phi} - \rho J(K, A, H, \bar{S}) + J_H H(b - \delta_H) - J_S \theta_i \bar{S} \\
&+ \left(\frac{1-\gamma}{\gamma}\right) \gamma^{\frac{1}{1-\gamma}} b^{\frac{\gamma}{\gamma-1}} J_K^{\frac{1}{1-\gamma}} J_H^{\frac{\gamma}{\gamma-1}} A^{\frac{\alpha}{1-\gamma}} K^{\frac{\beta}{1-\gamma}} (\vartheta_i \bar{S})^{\frac{1-\alpha-\beta-\gamma}{1-\gamma}} + \mu A J_A + \frac{\sigma_a^2 J_{AA} A^2}{2} \\
&+ \frac{\sigma_s^2 J_{SS} \bar{S}^2}{2} + J_{AS} \eta \sigma_a \sigma_s A \bar{S}.
\end{aligned} \tag{3.8}$$

With this maximized HJB equation (3.8), out task now is to *guess and verify* the closed-form representation of the value function $J(K, A, H, \bar{S})$. The analytical results are as follows:

Theorem 3.1. *Define*

$$\Theta \equiv \frac{\sigma_a^2 \alpha(1-\alpha) + \sigma_s^2 (\alpha+\beta+\gamma)(1-\alpha-\beta-\gamma)}{2} - \alpha\eta\sigma_a\sigma_s(1-\alpha-\beta-\gamma).$$

When $\phi = \beta$, we can find the closed-form representation of the value function (that satisfies both the HJB equation and the transversality condition, or TVC) of the form

$$J(K, A, H, \bar{S}) = \mathbb{X} K^{1-\beta} + \mathbb{Y} A^{\alpha} H^{\gamma} \bar{S}^{1-\alpha-\beta-\gamma} + \mathbb{Z}, \tag{3.9}$$

where

$$\mathbb{X} \equiv \frac{1}{1-\beta} \left(\frac{\beta}{\rho + (1-\beta)\delta_K}\right)^{\beta},$$

$$\mathbb{Y} \equiv \frac{\vartheta_i^{1-\alpha-\beta-\gamma}}{b^{\gamma}} \left(\frac{\beta}{\rho + (1-\beta)\delta_K}\right)^{\beta} \left(\frac{1-\gamma}{\rho - \alpha\mu - \gamma(b-\delta_H) + \vartheta_i(1-\alpha-\beta-\gamma) + \Theta}\right)^{1-\gamma}, \tag{3.10}$$

and

$$\mathbb{Z} \equiv -\frac{1}{\rho(1-\beta)}.$$

Moreover, the expressions for control variables are

$$C = \frac{\rho + (1-\beta)\delta_K}{\beta} K, \tag{3.11}$$

and

$$u = \frac{\rho - \alpha\mu - \gamma(b - \delta_H) + \vartheta_i(1 - \alpha - \beta - \gamma) + \Theta}{b(1-\gamma)}. \tag{3.12}$$

Proof. See Appendix A[1]

3.1.4 Comments on Theorem

I in turn comment on the main points in Theorem 3.1.

3.1.4.1 Value Function

Eq. (3.9) is the closed-form representation of the value function that will be used in the welfare analysis below. We can see that physical capital and the product of technology, human capital, and exhaustible resources are separable. Note that, again, when $\sigma_s = 0$ and $\eta = 0$, the value function (3.9) coincides with that of Hiraguchi (2013, Eq. 29). The non-separability here implies that endogenous growth comes from a *fusion* of technological progress, the accumulation of human capital, and use of global exhaustible resources.

3.1.4.2 Control Variables

Eq. (3.11) tells us that a consumption-capital ratio is constant. Note that the expression (3.11) completely coincides with Eq. (9) of Smith (2007), Eq. (11) of Marsiglio and La Torre (2012a), Eq. (30) of Hiraguchi (2013), and that presented in Proposition 2 of Hiraguchi (2014). It seems a bit at odds that the optimal level of consumption depends only on K, not on the rest of three state variables. Moreover, it is irrelevant to the uncertainty term such as σ_s. Wälde (2011a, Table

[1] The conditions for $u \in (0, 1)$ are complicated and can easily be obtained by straightforward calculation.

1) and Hiraguchi (2013) also observe this sort of property. Since it is found in the one-sector stochastic growth model of Smith (2007) as well, the optimal level of consumption appears to linearly and solely depend on physical capital stock. Indeed, Xie (1994, Lemma 2) already proves that, when $\phi = \beta$ in the deterministic Uzawa-Lucas model, the aggregate consumption C along any equilibrium path is *always* proportional to the physical capital stock K. Therefore, the finding here is rather positive: the finding of Xie (1994) carries over to the *stochastic* setting as well.[1]

These findings notwithstanding, we wish optimal consumption C dependent not only on K but also other state variables. In his survey on stochastic growth models that can be solved analytically, however, Wälde (2011a, p.621) concludes that "If optimal consumption is proportional to capital...but independent of total factor productivity, properties of optimal consumption are bound to be empirically questionable. The way out of this dilemma - analytically tractable closed-form solutions on the one hand and empirical relevance on the other - seems to be provided by closed-form solutions for models with parameter restrictions." Thus, finding out the economic reason why C is only dependent on K (or perhaps not) is the important question of the literature on stochastic growth models with closed-form solutions. Though such an analysis is of paramount importance, it is not the focus of this paper and is left for future research.

Eq. (3.12) says that the time spent in working is constant as well, again consistent with Hiraguchi (2013). You can see that u involves key parameters relevant to stochastic processes. Here, the most important difference between this paper and Hiraguchi (2013) is that, the effect of diffusion coefficients σ_a and σ_s on u is *indeterminate*.[2] Specifically, in Hiraguchi (2013, p.137), u is *always* increasing in σ_a. In sharp contrast to the previous studies, however, as there are *two* diffusion processes that are *correlated*, the effects of one shock depend on the other, and

[1] See also Smith (2007, footnote 6). He proves that, when $\phi = \beta$, consumption and physical capital grows at the same rate, so that their ratio is constant, resulting in (3.11).

[2] Concretely, we have $\partial u/\partial \sigma_s > 0$ as long as the inequality

$$\sigma_s > \frac{\alpha\eta}{\alpha + \beta + \gamma}\sigma_a,$$

holds, while we have $\partial u/\partial \sigma_s < 0$ when this inequality is reversed. In other words, when $\eta \neq 0$, the sign of $\partial u/\partial \sigma_s$ gets ambiguous.

can be indeterminate.

3.1.5 Growth and Welfare

The above observation has remarkable implications for the growth rate of human capital and welfare. In fact, one can show that, from the deterministic differential equation (3.1) and time spent in working (3.12), the growth rate of human capital $\mathcal{G}^H$ is given by

$$\mathcal{G}^H \equiv \frac{\dot{H}(t)}{H(t)} = \frac{b - \rho - \delta_H + \alpha\mu - \vartheta_i(1 - \alpha - \beta - \gamma) - \Theta}{1 - \gamma},$$

where $\dot{H} \equiv dH(t)/dt$. First, note that, in the absence of technological progress ($\mu = 0$), depreciation ($\delta_H = 0$), global resource sharing ($\vartheta_i = 0$), and uncertainty terms, the sign of $\mathcal{G}^H$ depends exclusively on the relative size of the efficiency parameter of human capital accumulation b and the subjective discount rate of households ρ.[1]

As the seminal paper of Barlevy (2004) shows, growth rates have close ties with welfare. In consequence, I discuss the growth and welfare implications *in parallel*. For instance, as we have the closed-form representation of the value function (3.9) and that of growth rate of human capital, one can analytically confirm that *what accelerates growth rate of human capital is absolutely welfare-improving*, and vice versa.

Since η is one of the most important parameters in this paper, it would be instructive to first understand its impact on growth rate of human capital. One can show that

$$\frac{\partial \mathcal{G}^H}{\partial \eta} > 0,$$

thus, higher correlation raises the growth rate of human capital (and improves welfare J) in country i. To see why, notice that the proportion of time devoted to learning u is decreasing in η, as shown in (3.12). This means that higher η discourages people to work, or equivalently, encourages them to accumulate

[1] As Kuwahara (2017) discusses in detail, this is the standard property of the deterministic Uzawa-Lucas model. It turns out that my model also has that property.

their new human capital. Therefore, since the accumulation of human capital is accelerated, the growth rate of human capital increases in response to higher correlation between two stochastic processes.[1] As this is the important mechanism through which parameters affect growth rate of human capital and welfare (most of analyses below can be understood via this channel), it would be worth illustrating.

Fig. 3.3 displays the relationship between the degree of correlation η and time allocation u (the left panel) and welfare J (or equivalently, growth rate of human capital, the right panel)[2]. The left panel shows that, as I have just explained, the higher correlation lowers the proportion of time devoted to working, and increases that devoted to learning. As it accelerates the accumulation of human capital, its growth rate is raised. Moreover, since the value function J is the function of state variables, more H improves welfare (the right panel). From this illustration, you can visually see that time allocation is the key to understanding the implications for growth rate of human capital, and hence welfare.

Next, what about the impact of resource shock σ_s on $\mathcal{G}^H$? Unlike parameters already discussed above, the sign is not determinate:

$$\frac{\partial \mathcal{G}^H}{\partial \sigma_s} \gtrless 0 \Leftrightarrow \sigma_s \lessgtr \frac{\alpha\eta}{\alpha+\beta+\gamma}\sigma_a. \tag{3.13}$$

In other words, the effects of higher resource uncertainty on the growth rate of human capital (and welfare) are *ambiguous*. To understand this point, see Fig. 3.4. It displays the relationship between between the size of resource shocks σ_s and welfare J. Each line presented is indexed by the correlation coefficient η.

We begin with the benchmark case of no correlation $\eta = 0$. In this case, as represented by the dashed line, higher uncertainty reduces welfare. To see why, note that u is *unambiguously* increasing in σ_s (since there is no correlation between stochastic processes). This means that higher uncertainty discourages people to

[1] By the same token, one can immediately see that $\mathcal{G}^H$ is raised by the more efficient accumulation of human capital ($\partial\mathcal{G}^H/\partial b > 0$) and higher growth rate of technology ($\partial\mathcal{G}^H/\partial\mu > 0$), while it is reduced by the higher depreciation rate of human capital ($\partial\mathcal{G}^H/\partial\delta_H < 0$) and lower share of resource ($\partial\mathcal{G}^H/\partial\vartheta_i < 0$).

[2] In the spirit of Mankiw et al (1992, p.432), I set $\alpha = \beta = \gamma = 0.3$. $b = 0.11$ is the value when Barro and Sara-i-Martin (2004) use in simulating the Uzawa-Lucas model. I choose $\mu = 0.02$ and $\delta_K = \delta_H = 0.03$ again following Mankiw et al. (1992). $\vartheta_i = 0.01$ and $\sigma_s = \sigma_a = 0.01$ are chosen purely for the illustrative purpose. Finally, following Caballé and Santos (1993) and Moll (2014), I set $\rho = 0.05$. MATLAB code is available at `http://dx.doi.org/10.17632/z856trfgvz.1`

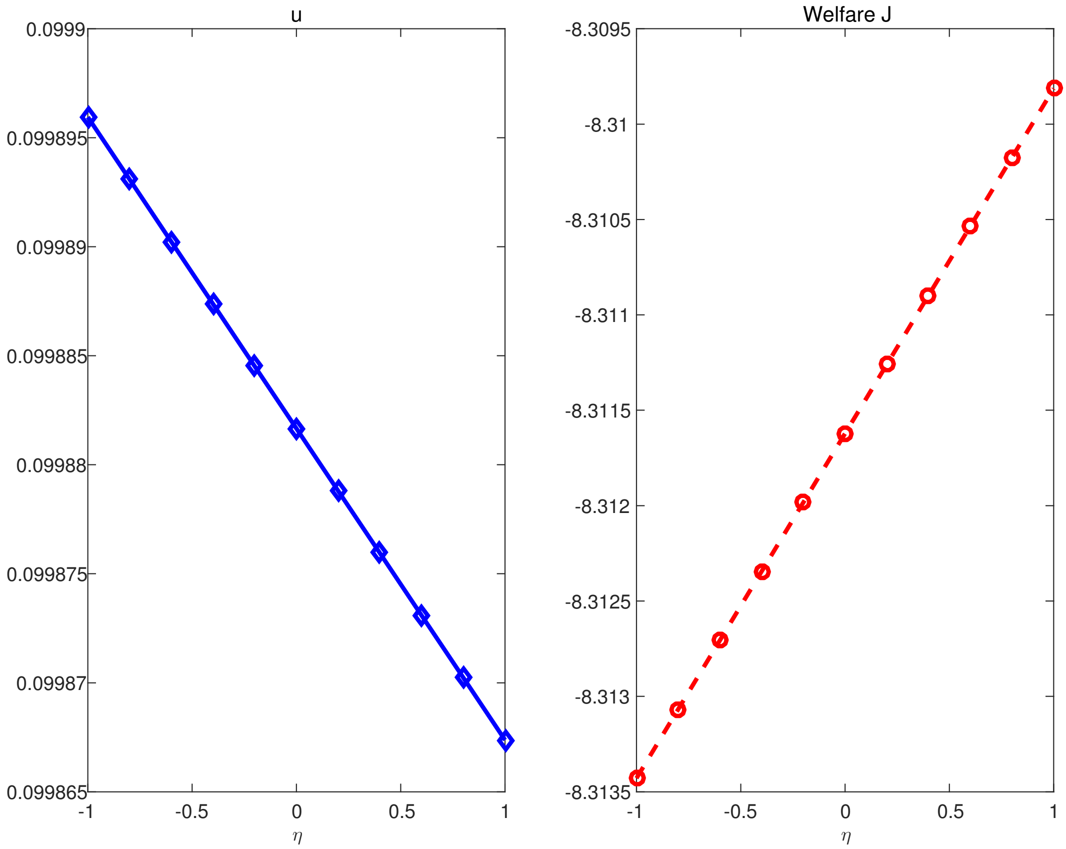

Figure 3.3 The relationship between correlation of two stochastic processes η and time allocation u, and welfare J.

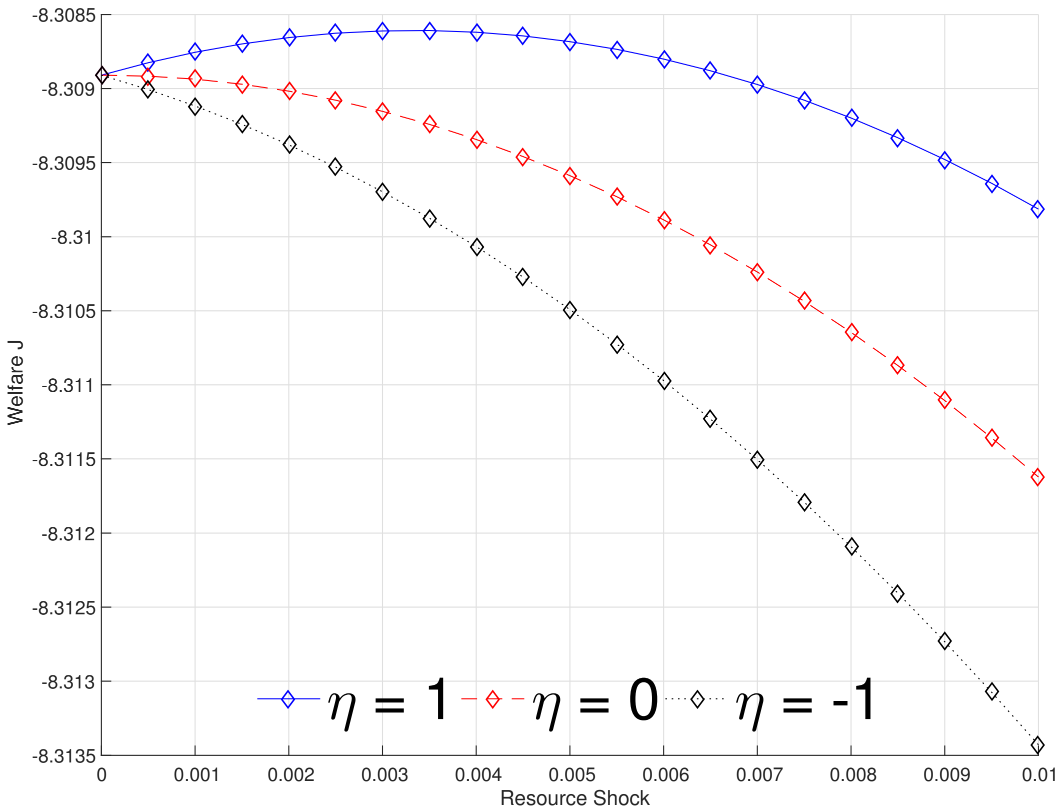

Figure 3.4 The relationship between the size of resource shocks σ_s and welfare J.

accumulate their new human capital. This realizes as output is lost due to higher uncertainty in the production sector. To compensate for this loss, people spend more time in the production sector, leading to human capital contraction. Then, since the stock of human capital decreases, the growth rate of human capital is decreased, and welfare is deteriorated. The case of negative correlation $\eta < 0$ can be interpreted in a similar way.

The interesting case would be when $\eta > 0$; that is, when two stochastic processes are positively correlated. In this case, we can see a *hump-shaped* relationship between welfare and the size of resource shocks. To understand this, remember that, as we saw above, the higher correlation raises the growth rate $\mathcal{G}^H$ and improves welfare J, as it leads to more human capital accumulation. Thus, in this case, there are two conflicting forces − the "accumulation" effect due to higher correlation and the "contraction" effect due to higher uncertainty. For a moderate degree of resource uncertainty, the former effect outweighs the latter, hence the net result is the *accumulation* of human capital, which raises its growth rate and improves welfare J. Beyond the threshold value at which the equality $\sigma_s = (\alpha\eta/(\alpha+\beta+\gamma))\sigma_a$ holds, however, the latter outweighs the former, resulting in the *contraction* of human capital. As such, the threshold value derived analytically is the point where the relative "power" of two conflicting forces switches, thereby yielding a hump-shaped relationship between welfare and uncertainty.

Although this point may be hard to swallow, another interpretation is to observe that, in response to higher resource uncertainty, households tend to spend more time in learning, thereby accelerating the accumulation of human capital. For a moderate degree of uncertainty, this positive effect of uncertainty dominates its standard negative impact due to risk aversion, thus leading to a net welfare *gain*. This is the reason why we see the positive relationship between uncertainty and welfare, as in real business cycle (RBC) models of Cho et al. (2015) and Lester et al. (2014), and the continuous-time stochastic AK model of Xu (2017). When shocks to resources are large enough, however, the usual negative effects outbalance the positive impact, resulting in a net welfare *loss*. As a consequence, there exists a hump-shaped relationship between uncertainty and welfare.

As this point is one of the central themes of this chapter, let me provide the further intuition underlying this finding. Remember that, as explained above, $\eta > 0$ is the case of resource-saving technological progress. At a first glance, higher uncertainty seems to reduce the willingness of firms to hire and invest, resulting in lower growth and hence deteriorating welfare. To recap, however, as Bloom (2014) discusses, higher uncertainty can possibly *stimulate* R&D because some firms may be *more* willing to innovate in preparation for a more uncertain future. Indeed, using the sectoral level data among 47 countries, Imbs (2007) empirically shows that investment tends to be stronger in response to higher uncertainty.

With this in mind, consider what would happen when uncertainty gets higher. In response to higher uncertainty, as in Bloom (2014), the incentive of firms to innovate can be stronger, thereby leading to technological progress. Then, remember the logic of the celebrated Romer (1990) model: human capital H is an *input* that can be used to increase the stock of technology A. Therefore, if firms wish to innovate in the wake of a more uncertain future, they need the "educated" (or *skilled*) labors (or researchers) for R&D, which is impossible with *unskilled* labors. In other words, there is now *higher* demand for human capital (and *higher* returns from human capital accumulation through a general equilibrium effect) in the aggregate economy. As a result, in reaction to the changed need of firms, households recognize the importance of increasing the stock of human capital H in the economy. As such, they tend to spend more of their time u in learning in a human capital sector, and this change of time allocation leads to the accumulation of human capital, resulting in higher growth and hence improving the welfare of agents.[1]

As long as uncertainty is moderate, this "positive" effect dominates the standard "negative" effect arising from higher uncertainty. In contrast, when uncertainty gets much higher, the mechanism described above is outbalanced by "negative" effects. In sum, there exists tensions between these two channels. The net outcome depends on their relative power, that is, on the one hand, the relationship between growth/welfare and uncertainty is *positive* when uncertainty is moderate, but on the other, it is *negative* when uncertainty is much higher. The combined force finally yields an inverted-U or a hump-shaped pattern that we see in Fig. 3.4,

[1] Indeed, Bretschger (2005, p.159) stresses the importance of considering the relationship between education of researchers and their productivity in research.

which is completely consistent with the empirical finding of García-Herrero and Vilarrubia (2007): they find a *Laffer curve* between growth and uncertainty, and conclude that a moderate degree of uncertainty can be growth-enhancing while very high uncertainty is clearly detrimental.

What about technology shocks? Note that, since we have

$$\frac{\partial \mathcal{G}^H}{\partial \sigma_a} \gtrless 0 \Leftrightarrow \sigma_a \lessgtr \frac{\eta(1-\alpha-\beta-\gamma)}{1-\alpha}\sigma_s, \tag{3.14}$$

one can easily gauge that the impact of technology shocks on the growth rate of human capital and welfare is again *ambiguous*, and that we would see the same patterns described in Fig. 3.4. As you can see in Fig. 3.5, the underlying mechanism through which the hump-shaped pattern emerges is completely the same with that for resource shocks. Therefore, I refrain from repeating the same explanation above.

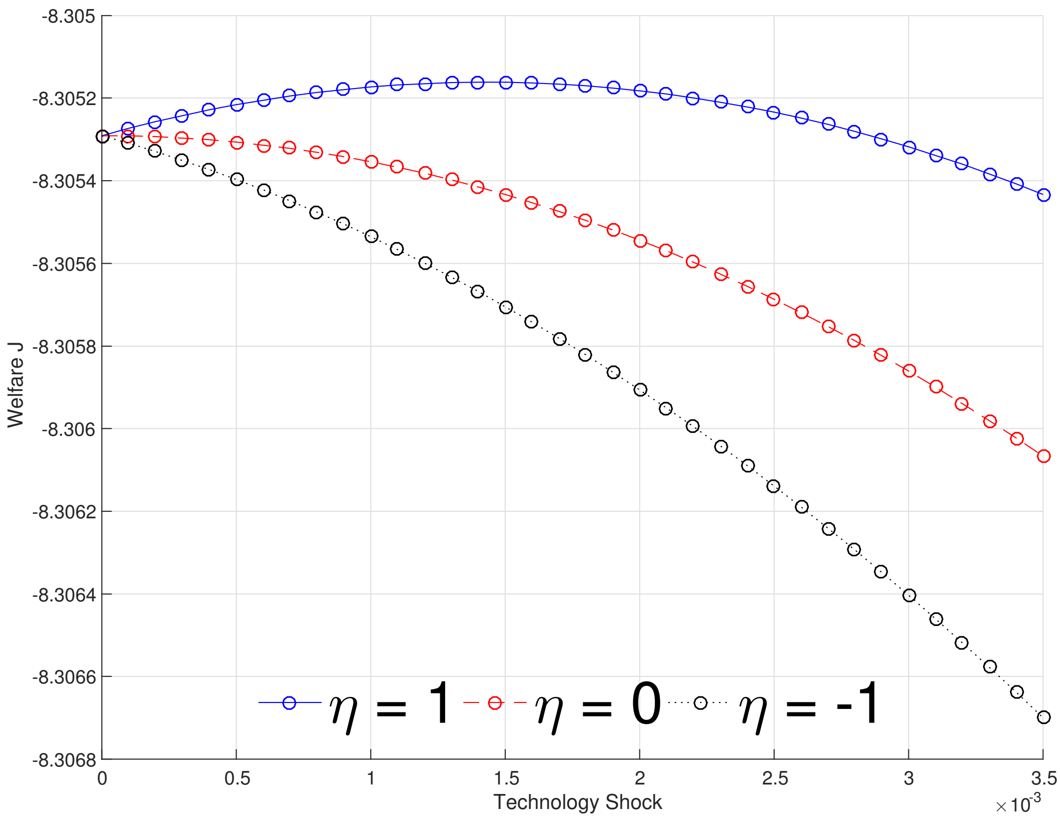

Figure 3.5 The relationship between the size of technology shocks σ_a and welfare J.

3.1.6 Empirical Evidence and Simulation

The finding that higher uncertainty can accelerate economic growth may look counterintuitive and unrealistic. Although there are some reasons to believe that there exists a *positive* link between growth and uncertainty (for example, in addition to the R&D argument in Bloom (2014), we can think of the precautionary savings channel; higher uncertainty raises a savings rate, and hence a higher investment rate. This stronger investment leads to higher growth in the long run), we need to examine some *empirical* evidence to assess whether the above *theoretical* findings are possible in reality. In what follows, I provide some empirical evidence on resource uncertainty and some simulation.

3.1.6.1 Empirical Evidence on Resource Uncertainty

Are there any relevant episodes that demonstrate whether resource uncertainty matters in reality? Carefully examining three case studies in Poland, the recent paper of Lis and Stasik (2017) shows that resource uncertainty arises from the lack of data on quality, quantity, and the location of the resource. Specifically, they obtain the audio and/or video recordings and transcripts of public meetings on a shale gas exploration in Poland. They then analyze them to examine communication between participants (such as local people) and those who wish to promote a shale gas exploration (such as community representatives and geologists). From the recorded communications, they find that uncertainty arising from the difficulty in collecting accurate information on resources prevents them from understanding each other.

Based on these, Lis and Stasik (2017, p.31) concludes that "uncertainty both about the existence of the resource and the mode of its existence as being exploitable and economically viable or useless in the state of technological development and the market situation of that time" has been created in Poland. In addition, estimating the supply cost curves (of oil and gas) in Europe, Aguilera and Ripple (2012, p.389) claim that "...though the past is not always an indication of the future, history would suggest that producers may develop the technologies needed to offset the cost-increasing effects of...oil and gas resources." Therefore, these empirical evidence from Poland and Europe suggest the importance of resource uncertainty and the validity of the firm channel discussed in Bloom (2014) and confirmed in Imbs (2007).

3.1.6.2 Simulation

So far, I have stressed the importance of stochastic elements in light of theoretical and empirical findings. One may, however, still wonder why my analysis needs uncertainty; why do we need *stochastic* differential equations (3.3) and (3.4) that are mathematically more involved than *deterministic* ordinary differential equations? How does the stochastic environment compare better with the deterministic environment, if indeed? To clarify these points, though less informal than the empirical analysis with respect to Ramey and Ramey (1995), I simulate two stochastic differential equations to justify their use.

To begin with, the solution to (3.3) is (see Chang (2004) or Appendix 2.C):

$$A_i(t) = A_i(0)e^{\left(\mu - \frac{\sigma_a^2}{2}\right)t + \sigma_a z_a(t)}, \tag{3.15}$$

while that to (3.4) is

$$\bar{S}(t) = \bar{S}(0)e^{\left(-\vartheta_i - \frac{\sigma_s^2}{2}\right)t + \sigma_s z_s(t)}. \tag{3.16}$$

With these analytical solutions, by setting initial values $A(0) = \bar{S}(0) = 1$ for brevity and using parameters used above, we can simulate (3.15) and (3.16). The results are displayed in Fig. 3.6. First, we begin with the deterministic environment $\sigma_a = \sigma_s = 0$. The upward line represents the evolution of $A(t)$, whereas the downward line represents that of $S(t)$. In this case, note that, for all t, we have $A(t) > S(t)$. So, the stock of technology is always larger than that of exhaustible resources − the prediction deterministic models would make. When agents know that technological progress will go on in the future, there is little or no incentive for them to change their behavior, even if the resource depletes at a constant rate. As such, in the deterministic setup, we won't observe the U-shaped pattern.

Next, we move on to the stochastic environment. According to Eq. (3.13), the condition for $\mathcal{G}^H > 0$ is $\sigma_s < (\alpha\eta/(\alpha + \beta + \gamma))\sigma_a$. Under the perfectly positive

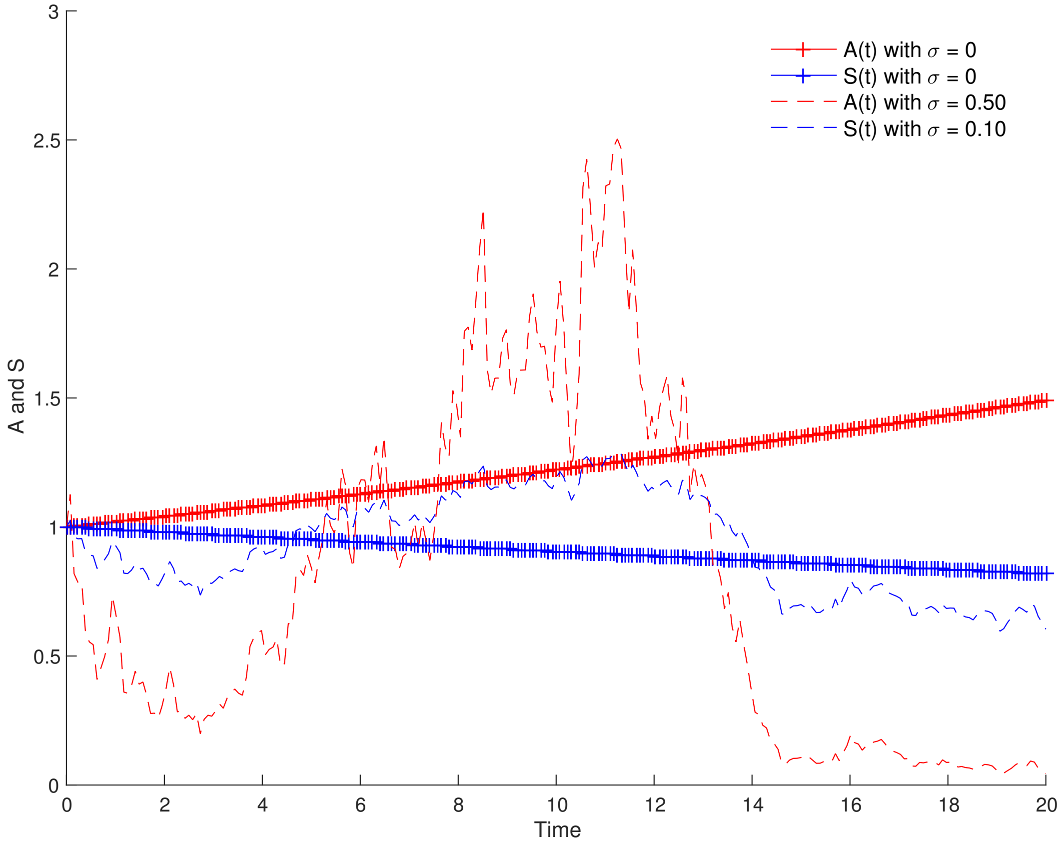

Figure 3.6 Simulation of two stochastic differential equations (3.3) and (3.4).

correlation $\eta = 1$ and assumed parameter values, it boils down to $\sigma_a > 3\sigma_s$. Then, by choosing $\sigma_a = 0.50$ and $\sigma_s = 0.10$ to satisfy this inequality, we can get some *quantitative* clue to the emergence of a U-shaped pattern from Fig. 3.6. In this case, we can see that the growth path of technology and resources fluctuates around their long-run trend with $\sigma_a = 0$ or $\sigma_s = 0$. What is important in this scenario is that we sometimes have $A(t) < S(t)$; the stock of technology is *less* than that of resources (for instance, between $t = 0$ and $t = 5$) $-$ the prediction only *stochastic* models can make.

This observation is likely to verify the argument of Bloom (2014). If $A(t) < S(t)$, in the face of a more uncertain future, firms anticipate that they need to innovate to overcome the resource scarcity. As long as $A(t) < S(t)$, their concern does not disappear, and they are always under threat of resource scarcity or depletion. *This* can stimulate the R&D of firms, leading to technological progress and resulting in higher growth, and hence a *positive* relationship between growth and higher uncertainty (as long as it is moderate). This process is likely to continue unless $A(t) = S(t)$ or $A(t) > S(t)$ is achieved. To sum up, our simulation confirms that uncertainty generates a situation in which firms are more willing to innovate, as the stock of technology can be less than that of resources. Thus, the emergence of a U-shaped pattern may not only theoretically, but also quantitatively, be possible.

Having provided some empirical support, the findings of this section can be summarized as follows:

Proposition 3.1. *I find the closed-form solution to the stochastic Uzawa-Lucas model in which both technological progress and the depletion of exhaustible resources are driven by a correlated Brownian motion process. The higher correlation between two stochastic processes always raises the growth rate of human capital and improves welfare. When two stochastic processes are positively correlated, there exists a U-shaped relationship between resource or technology shocks and growth (and welfare), as long as they are moderate.*

Therefore, in most cases, shortages of natural resources really constrain economic growth and deteriorate welfare. In contrast, when resource dynamics and technological progress positively interact under low uncertainty, resource scarcity may *not* constrain economic growth and reduce economic welfare.

3.2 Concluding Remarks

To summarize for the purposes of policy implications, technological progress is welcome, as it is the ultimate engine of growth and probably improves welfare. From the resource scarcity viewpoint, however, it is just a sufficient condition; not all types of technological progress are resource-saving. Moreover, we cannot precisely know whether the arrival of resource-saving technological progress will repeat in the future. Thus, the answer to the above question is not definitive; at the *global* level, if we can promote the resource-saving technological progress and make it arrive more frequently, we may overcome the resource scarcity. In contrast, if we fail to promote the resource-saving technological progress, or if its arrival process fails to repeat in the future, resource scarcity may pose a serious threat to economic growth, and to the welfare of humanity. To avoid these, a policy should be designed so that uncertainty is reduced (as long as the degree of uncertainty is very high), and that the invention of resource-saving technology is encouraged.

I would like to emphasize that, when the degree of uncertainty is moderate, it may *accelerate* economic growth and *improve* welfare. This is the central policy implication of this chapter: it cannot be figured out in the *deterministic* endogenous growth model of Vita (2007), on which this chapter builds. It is possible only by considering the *stochastic* elements, as in Dasgupta and Heal (1974), Clark (1979), and Pindyck (1980, 1984) at the cost of intractability. As I discussed above, when the future gets more uncertain, firms may be *more* willing to innovate (Bloom, 2014). This uncertainty-induced technological progress may strengthen economic growth and improve welfare, if it is resource-saving.

To achieve these goals, the government first needs to correctly measure the degree of uncertainty prevailing in the economy. This allows it to assess whether the policy must be oriented to reducing uncertainty or not. Moreover, the government should collect as much information on firms as possible, so that it can know how many firms in the economy have sufficient willpower to innovate in the face of higher uncertainty. Or, if the cost of collecting information is very high, from the institutional viewpoint, the government can alternatively implement the educational reforms to foster the entrepreneurship of students who will be the potential innovators. In this sense, my analysis of growth and resources in the framework of Uzawa (1965) and Lucas (1988) seems appropriate.

3.A Guide to Analytical Solutions

This appendix briefly describes how to find the closed-form representation of the value function (3.9) in Theorem 3.1. For this purpose, postulate the tentative value function of the form

$$J(K, A, H, \bar{S}) = \mathbb{X}K^{\theta_1} + \mathbb{Y}H^{\theta_2}A^{\theta_3}\bar{S}^{\theta_4} + \mathbb{Z},$$

where $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, θ_1, θ_2, θ_3, and θ_4 are all unknown constants to be determined. The relevant partials are $J_K = \mathbb{X}\theta_1 K^{\theta_1-1}$, $J_{KK} = \mathbb{X}\theta_1(\theta_1 - 1)K^{\theta_1-2}$, $J_H = \mathbb{Y}\theta_2 H^{\theta_2-1}A^{\theta_3}\bar{S}^{\theta_4}$, $J_A = \mathbb{Y}\theta_3 H^{\theta_2}A^{\theta_3-1}\bar{S}^{\theta_4}$, $J_{AA} = \mathbb{Y}\theta_3(\theta_3 - 1)H^{\theta_2}A^{\theta_3-2}\bar{S}^{\theta_4}$, $J_S = \mathbb{Y}\theta_4 H^{\theta_2}A^{\theta_3}\bar{S}^{\theta_4-1}$, $J_{SS} = \mathbb{Y}\theta_4(\theta_4 - 1)H^{\theta_2}A^{\theta_3}S^{\theta_4-2}$, and $J_{AS} = \mathbb{Y}\theta_3\theta_4 H^{\theta_2}A^{\theta_3-1}\bar{S}^{\theta_4-1}$. To obtain the explicit expression, substitute these partials into the maximized HJB equation (3.8). Then, set $\theta_1 = 1 - \beta$, $\theta_2 = \gamma$, $\theta_3 = \alpha$, and $\theta_4 = 1 - \alpha - \beta - \gamma$. Finally, imposing the parameter restriction $\phi = \beta$, you can find the explicit expressions for $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$, and consequently, those for control variables C and u and for the value function $J(K, A, H, \bar{S})$ in Theorem 3.1. Moreover, one can establish that the value function J satisfies the optimality conditions. For Brownian uncertainty, the proof requires the verification theorem; see Chang (2004), Hiraguchi (2013), or Appendix 2.B.

3.B Supplementary Data

Fig. 3.7 displays the annual level of U.S. crude oil reserves from 1900 to 2014 (values shown in millions of barrels; data are from Macrotrends[1]). Until 1970, they had steadily increased: in 1970, U.S. crude oil reserves were approximately at 39,000 millions of barrels (compared with 2,900 millions in 1900). Since the 1970s energy crisis, however, the level of reserves has decreased. In 2008, it was approximately at 19,100 millions.

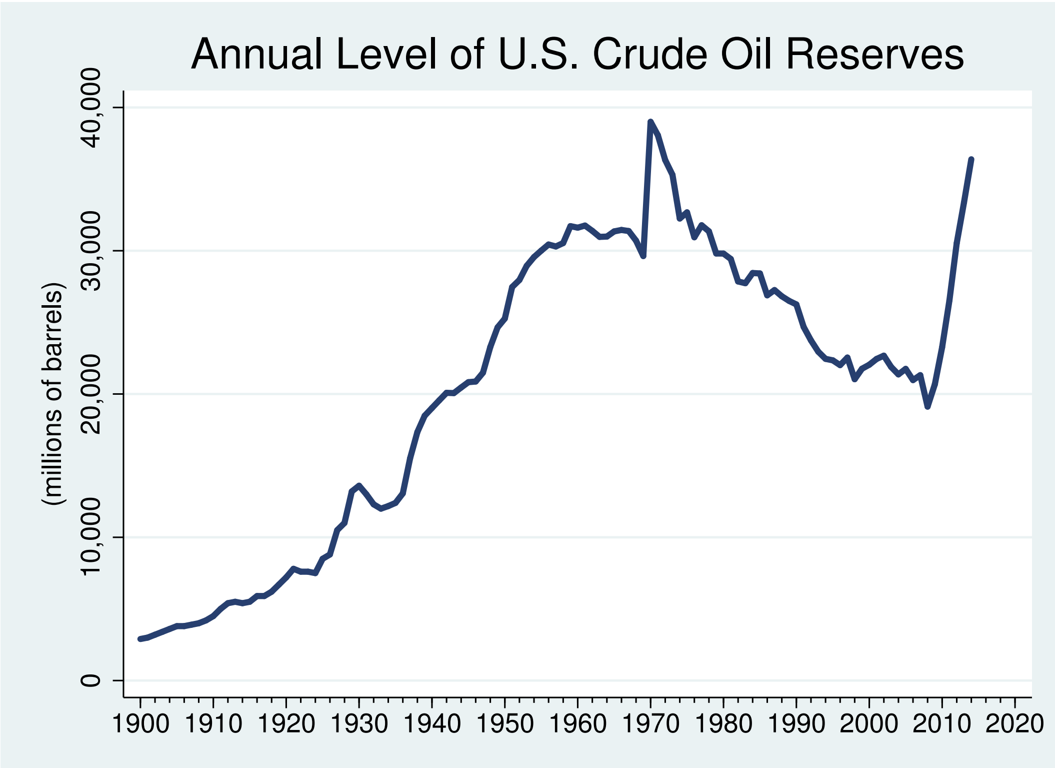

Figure 3.7 Annual level of U.S. crude oil reserves.

Somewhat surprisingly, since 2008, the level of U.S. crude oil reserves has continued the *upward* trend: in 2014, they were approximately at 36,400 millions of barrels. Though we don't know whether this upward trend will continue, together with Fig. 3.2, we have a puzzle – prices are going *up*, and quantities are also *increasing*. So, whether we are using up exhaustible resources are uncertain.

I use the data for Fig. 3.7 to produce Fig. 3.8. Just like in Chapter 1, it shows the relationship between economic growth and resource uncertainty. The link seems negative with the correlation coefficient of −0.18, though it is insignificant

[1] `https://www.macrotrends.net/2565/us-crude-oil-reserves-historical-chart`

with the p-value of 0.58. Although a linear link may be spurious, an inverted-U association may not. It seems consistent with an inverted-U shaped relation between growth and uncertainty shown in a theoretical part of this chapter.

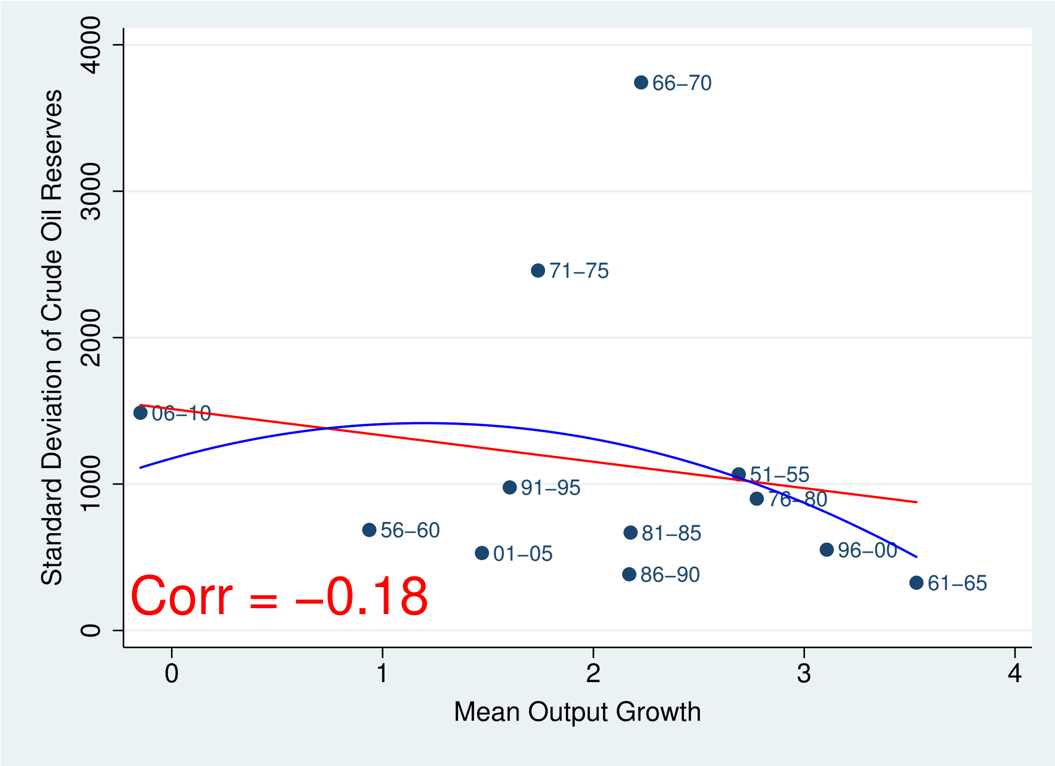

Figure 3.8 Growth and Resource Uncertainty between 1951 and 2010.

Chapter 4

Consumption, Welfare, and Stochastic Population Dynamics When Technology Shocks Are (Un)tied

4.1 Introduction

Many economic decisions involve uncertainty about the outcome of the choice we make.[1] We cannot know the final result of the decision until it occurs. When uncertainty affects households' decisions, what impact does it have on the economy as a whole? The standard narrative would be as follows: the level of consumption is reduced due to a precautionary saving motive, and subsequently, the welfare of households is deteriorated.

Recent several studies, however, find this intuitive narrative *not* necessarily true. Furthermore, empirical studies of Bloom (2009) and Bachmann et al. (2013) demonstrate that higher uncertainty has sizable impacts on the macroeconomy. As consumption – the source of welfare – is the most important variable in economics, it is necessary to revisit the standard narrative. It will help policymakers implement the optimal policy that can maximize growth and welfare.[2]

[1] This chapter is based on Tsuboi (2019a).

[2] The uncertainty-welfare nexus is not reviewed at all even in a survey of Bloom (2014) on uncertainty.

Specifically, first, the findings from stochastic growth literature suggest that, in theory, impacts of higher uncertainty on consumption isn't conclusive. Although quantitative macroeconomic studies typically find that consumption exhibits the hump-shaped pattern in response to higher uncertainty, Marsiglio and La Torre (2012b) and Hiraguchi (2013) show that larger demographic or technology shock have *nothing* to do with the optimal level of consumption. In a similar framework, however, Bucci et al. (2011) and Marsiglio and La Torre (2012a) show that, in response to them, the optimal level of consumption is unambiguously *reduced*.

Second, focusing on the *household* side, several recent studies of Cho et al. (2015), Lester et al. (2014), and Xu (2017) find that *higher* uncertainty may *improve* the welfare of agents, because purposeful agents may make use of uncertainty in their favor, under some conditions. This is in sharp contrast to the standard narrative that presumes the complete absence of uncertainty for the welfare-maximizing outcome. If they are true, the conventional intuition that uncertainty must, at any rate, completely be wiped out (so that welfare ameliorates), can be misleading for the design of optimal policy. As such, we need to radically understand which shocks probably affect the optimal level of consumption, and under what condition(s) higher uncertainty may improve the welfare of agents.

To complement the studies cited above, I construct the stochastic two-sector optimal growth model of Uzawa (1965) and Lucas (1988) in which both population dynamics and technological progress are driven by the *correlated* Brownian motion process (thereby focusing on the *production* side). Imposing one parameter restriction of Xie (1991; 1994), I first find the closed-form solution to that model. Analytical solutions are more advantageous to the theoretical analysis than numerical simulation in most cases; they allow us to inspect the underlying mechanism in the most transparent way. I then use that solution to characterize the behavior of agents in response to higher uncertainty. In the Uzawa-Lucas model, control variables are not only consumption, but also the allocation of time between two sectors: one is a production sector where people spend time producing final goods, while the other is a human capital sector where people spend time accumulating new human capital. With this formulation, as a consequence, agents have an *endogenous* choice − either to work in a production sector or to learn in a human capital sector. This may allow them to make use of higher uncertainty in their favor.

As a preview, in the baseline model in which only population dynamics is stochastic, I show that larger demographic shocks unambiguously reduce the optimal level of consumption because of a precautionary saving motive, as in the standard narrative. At the same time, I find that higher demographic uncertainty, on the other hand, are always welfare-*improving*. This takes place as agents tend to allocate more of their time in learning in a human capital sector, when they are ramified by higher uncertainty. Because this encourages further human capital accumulation, growth is accelerated and welfare is improved.

In the extended model in which stochastic population dynamics is tied to stochastic technological progress, I find that the results critically hinge on their *interaction*. When they are untied or positively tied, the qualitative implications of the baseline model remain unchanged. In contrast, when they are *negatively* tied, there emerges an *inverted U-shaped* relationship between uncertainty and consumption, and a *U-shaped* relationship between uncertainty and welfare. The interaction of two stochastic processes, which are absent in the previous studies, yields novel mechanisms through which consumption and welfare are perturbed by higher uncertainty. As such, this chapter derives some policy implications when thinking about the policy response to higher uncertainty about population dynamics and technological progress.

Summing up, the purpose of this chapter is to analytically characterize the relationship between uncertainty and the optimal level of consumption or welfare, by finding the closed-form solution to the stochastic Uzawa-Lucas model that features the *correlated* Brownian motion process.

This chapter is organized as follows. Sect. 2 sets up the baseline Uzawa-Lucas model with stochastic population dynamics. Sect. 3 introduces stochastic technological progress and examines the interaction between two diffusion processes. Concluding remarks appear in Sect. 4.

4.2 The Baseline Model

In this section, I develop the streamlined (but stochastic) Uzawa-Lucas model. It is a two-sector endogenous growth model in which human capital is an explicit input. Consider a closed economy in continuous time running to an infinite horizon. The economy is inhabited by a large number of households. I assume that all households are identical, so that the economy trivially admits a representative household. This means that the demand and supply side of the economy can be represented as if it resulted from the behavior of a single household.

A representative household is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u(t) \in (0, 1)$ denote the fraction of time spent working to produce final goods $Y(t)$. So, $1-u(t)$ represents the fraction of time spent learning. I assume that the amount of leisure is fixed exogenously, so there is no choice about it. The implicit assumption is that part of the human capital in this economy can be used for further human capital accumulation. Therefore, it essentially captures the technology of the economy to generate human capital, such as school system and training.

4.2.1 Production and Population Dynamics

I first begin with the model with stochastic population dynamics only. It seems that this is more instructive to illustrate the essentials, mechanisms, and implications of the model, than immediately scramble to the more general setting with two correlated stochastic processes. A representative firm has access to the Cobb-Douglas production function:

$$Y(t) = (u(t)H(t))^{\alpha} K(t)^{\beta} L(t)^{1-\alpha-\beta}, \tag{4.1}$$

where $H(t)$ is the aggregate stock of human capital. $K(t)$ is the aggregate stock of physical capital. $\alpha \in (0, 1)$ is the human capital share of income. $\beta \in (0, 1)$ is the physical capital share of income. $L(t)$ is the size of population (or raw labor). Implicit in (4.1) is that some members of a representative household are skilled labors (while others are unskilled or raw labors) so that those who are learning do not contribute to the production of final goods $Y(t)$. The initial stock of aggregate human and physical capital, $H(0) = H_0 > 0$ and $K(0) = K_0 > 0$, are both given.

I assume that population dynamics follows a geometric Brownian motion process:

$$dL(t) = nL(t)dt + \sigma_L L(t)dz_L(t), \tag{4.2}$$

where n is the exogenous rate of population growth. $dz_L(t)$ is the increment of a Brownian motion (or Wiener) process for population dynamics, such that the mean $\mathbb{E}(dz_L) = 0$ and variance $\mathcal{V}(dz_L) = dt$. In other words, changes in a Brownian motion process over any finite interval of time are normally distributed, with a variance that increases linearly with the time interval. As the Brownian motion process is nonstationary, its variance will go to infinity over the long run. $\sigma_L \geq 0$ is the diffusion coefficient of population; if $\sigma_L = 0$, then we would recover the deterministic limit. Eq. (4.2) is exactly Eq. (7) in the seminal work of Merton (1975, p.377). Bucci et al. (2011) and Hiraguchi (2013) assume $L = 1$ to simplify their analysis. I will show, however, that this assumption makes trouble in studying the impacts of demographic (and later, technology) shocks on the optimal level of consumption. I assume $L(0) = L_0 > 0$ so that $L(t) > 0$ for all t with probability 1.

The per capita equivalent of Eq. (4.1) is

$$y(t) = (u(t)h(t))^{\alpha}k(t)^{\beta}, \tag{4.3}$$

where $y(t) \equiv Y(t)/L(t)$, $k(t) \equiv K(t)/L(t)$, and $h(t) \equiv H(t)/L(t)$ respectively denote per capita output, physical capital, and human capital.

4.2.2 Capital Accumulation and Household

The accumulation of per capita human capital $h(t)$ is governed by the following controlled diffusion process:

$$dh(t) = \left(b(1 - u(t)) - (n + \delta_H - \sigma_L^2)\right) h(t)dt - \sigma_L h(t)dz_L(t), \tag{4.4}$$

where I use Itô's Lemma. $b > 0$ is an exogenous parameter that indicates how efficient the accumulation of human capital is. $\delta_H \in (0, 1)$ captures the depreciation rate of human capital, which comes about, for example, because new machines and techniques are introduced that erode the existing human capital of the worker (Acemoglu, 2009, p.363). I call Eq. (4.4) the *stochastic Uzawa-Lucas equation* in what follows.

In a similar vein, the stochastic law of motion for the economy-wide resource constraint is given by

$$dk(t) = \left(\underbrace{(u(t)h(t))^{\alpha} k(t)^{\beta}}_{\equiv y(t)} - (n + \delta_K - \sigma_L^2)k(t) - c(t) \right) dt - \sigma_L k(t) dz_L(t), \quad (4.5)$$

where I again use Itô's Lemma. $\delta_K \in (0, 1)$ is the depreciation rate of physical capital. $c(t) \equiv C(t)/L(t)$ denotes per capita consumption of final goods. $C(t)$ is aggregate consumption. You may note that Eq. (4.5) is exactly the *stochastic Solow equation* first derived by Merton (1975).

Preferences of a representative household are given by the standard constant relative risk aversion (CRRA) utility function:

$$\mathbb{E} \int_0^{\infty} e^{-\rho t} \frac{c(t)^{1-\phi} - 1}{1 - \phi} dt, \quad (4.6)$$

where $\mathbb{E}$ is the expectation operator with respect to the information set available to a representative household. $\rho > 0$ is the subjective discount rate; the rate at which utility is discounted. $\phi > 0$ is the index of relative risk aversion (and $1/\phi$ is the elasticity of intertemporal substitution). When future consumption is uncertain, a larger ϕ makes future utility gain smaller, raising the value of additional future consumption.

4.2.3 Stochastic optimization

A representative household maximizes its expected utility (4.6) subject to two stochastic processes (4.4) and (4.5). To solve this optimization problem, let $J(k, h)$ be a value function (or an indirect utility function). Then, the associated Hamilton-Jacobi-Bellman (HJB) equation reads:

$$\max_{\{c,u\}} \left(\frac{c^{1-\phi}-1}{1-\phi} - \rho J(k,h) + J_K \frac{dk}{dt} + J_H \frac{dh}{dt} + J_{KH} \frac{(dk)(dh)}{dt} + \frac{J_{KK}}{2}\frac{(dk)^2}{dt} + \frac{J_{HH}}{2}\frac{(dh)^2}{dt} \right),$$

that is,

$$\begin{aligned}&\max_{\{c,u\}} \left(\frac{c^{1-\phi}-1}{1-\phi} - \rho J(k,h) + J_K \left(y - (n+\delta_K - \sigma_L^2)k - c \right) + J_H h \left(b(1-u) - (n+\delta_H - \sigma_L^2) \right) \right. \\ &\quad \left. + J_{KH} k h \sigma_L^2 + \frac{J_{KK}\sigma_L^2}{2} k^2 + \frac{J_{HH}\sigma_L^2}{2} h^2 \right),\end{aligned} \tag{4.7}$$

where $J_K \equiv \partial J(k,h)/\partial k$, $J_H \equiv \partial J(k,h)/\partial h$, $J_{KH} \equiv \partial^2 J(k.h)/\partial k \partial h$, $J_{KK} \equiv \partial^2 J(k,h)/\partial k^2$ and $J_{HH} \equiv \partial^2 J(k,h)/\partial h^2$. First-order conditions are

$$c = J_k^{-\frac{1}{\phi}}, \tag{4.8}$$

and

$$u = \frac{k^{\frac{\beta}{1-\alpha}}}{h} \left(\frac{\alpha J_K}{b J_H} \right)^{\frac{1}{1-\alpha}}. \tag{4.9}$$

Substituting these first-order conditions (4.8) and (4.9) back to the HJB equation (4.7) and rearranging, we get

$$\begin{aligned}0 = &\frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} - \rho J(k,h) - J_K k(n+\delta_K - \sigma_L^2) + J_H h(b - n - \delta_H + \sigma_L^2) + \\ &J_{KH} k h \sigma_L^2 + \frac{\sigma_L^2}{2}\left(J_{KK} k^2 + J_{HH} h^2 \right) + \left(\frac{1-\alpha}{\alpha} \right) \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{1-\alpha}} k^{\frac{\beta}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}}.\end{aligned}$$

With this equation, our task now is to guess and verify the closed-form representation of a value function $J(k,h)$. With one parameter constraint, we can obtain a closed-form solution. It can be summarized as follows:

Theorem 4.1. *When $\phi = \beta$, there exists the closed-form representation of the value function that satisfies the HJB equation and the transversality condition (or*

TVC) of the form:

$$J(k,h) = \Omega_X k^{1-\beta} + \Omega_Y h^{\alpha} + \Omega_Z, \tag{4.10}$$

where

$$\Omega_X \equiv \frac{1}{1-\beta}\left(\frac{\beta}{\rho + (1-\beta)(n+\delta_K-\sigma_L^2) + \frac{\sigma_L^2}{2}\beta(1-\beta)}\right)^{\beta}, \tag{4.11}$$

$$\Omega_Y \equiv \frac{\Omega_X(1-\beta)}{b^{\alpha}}\left(\frac{1-\alpha}{\rho - \alpha(b-n-\delta_H+\sigma_L^2) + \frac{\sigma_L^2}{2}\alpha(1-\alpha)}\right)^{1-\alpha}, \tag{4.12}$$

and

$$\Omega_Z \equiv -\frac{1}{\rho(1-\beta)}. \tag{4.13}$$

Moreover, we can find the explicit expressions for two control variables:

$$c = \frac{\rho + (1-\beta)(n+\delta_K-\sigma_L^2) + \frac{\sigma_L^2}{2}\beta(1-\beta)}{\beta}k, \tag{4.14}$$

and

$$u = \frac{\rho - \alpha(b-n-\delta_H+\sigma_L^2) + \frac{\sigma_L^2}{2}\alpha(1-\alpha)}{b(1-\alpha)}. \tag{4.15}$$

Proof. See Appendix A[1].

4.2.4 Discussion

I in turn comment on key points in Theorem 4.1.

[1]The condition for $u \in (0,1)$ is

$$\alpha(b-n-\delta_H+\sigma_L^2) - \frac{\sigma_L^2}{2}\alpha(1-\alpha) < \rho < \alpha(b-n-\delta_H+\sigma_L^2) - \frac{\sigma_L^2}{2}\alpha(1-\alpha) + b(1-\alpha).$$

4.2.4.1 Control Variables and Welfare

Eq. (4.14) tells us that a consumption-capital ratio is constant. Note that it is *dependent* on the shock term:

$$\frac{\partial c}{\partial \sigma_L} < 0, \tag{4.16}$$

in other words, higher demographic uncertainty (such as war, invasion, or epidemic) reduces the optimal level of (per capita) consumption. This is because of a precautionary saving motive, as Bucci et al. (2011) explain in the context of technology shocks. I illustrate this point in Fig. 4.1. For this illustration, I use $\alpha = 1/3$, $\beta = 0.40$, $b = 0.11$, $\delta_K = \delta_H = 0.03$, $n = 0.01$, and $\rho = 0.05$.[1] The upper left panel shows that consumption falls in response to higher demographic uncertainty σ_L. A rise in uncertainty leads agents to increase their precautionary saving, which reduces their consumption expenditure (Bloom, 2014, p.165). As agents are so uncertain about the future, they choose to consume *less* and save *more*. As a result, higher demographic uncertainty reduces the optimal level of consumption. This finding is consistent with that of quantitative macroeconomic studies.

It turns out that the assumption $L = 1$ is problematic; though it simplifies the analysis, it eliminates the channel through which shocks affect the optimal level of (per capita) consumption, as in Hiraguchi (2013). Relaxing the assumption of $L = 1$, I can argue that demographic uncertainty in fact affects c.

Next, Eq. (4.15) says that the fraction of time spent working is constant, consistent with Hiraguchi (2013) and Tsuboi (2018). Since we are interested in what impact demographic shocks have on the allocation of time, notice that

$$\frac{\partial u}{\partial \sigma_L} < 0, \tag{4.17}$$

[1]Following Mankiw et al. (1992, p.432), I set the human capital share $\alpha = 1/3$. For physical capital share, it has been commonplace in macroeconomics to use 1/3. As Karabarbounis and Neiman (2014) document, however, the labor share is declining globally. Therefore, I set $\beta = 0.40$, the value used by Ahn et al. (2018). $b = 0.11$ is used by Barro and Sara-i-Martin (2004) in simulating the Uzawa-Lucas model. I choose the exogenous technological progress rate $\mu = 0.02$ (to be used in the next section) and $\delta_K = \delta_H = 0.03$, again following Mankiw et al. (1992). I use $n = 0.01$, which is close to the recent population growth rate in the U.S. or Asia. Finally, following Caballé and Santos (1993) and Moll (2014), I set $\rho = 0.05$. These parameter values are somewhat realistic, but the main point is to show *qualitatively* how the model works.

so, higher demographic uncertainty discourages people to work in a production sector (upper right panel). Or equivalently, it encourages people to accumulate their human capital in the other sector. Therefore, higher demographic uncertainty leads to human capital accumulation. This has important implications for welfare (also in the next section). To see why, first, note that

$$\frac{\partial J(k,h)}{\partial \sigma_L} > 0, \tag{4.18}$$

thus, higher demographic uncertainty unambiguously *improves* the welfare of agents (lower right panel). You can see that increases in σ_L lead to the stronger contribution of (per capita) human capital h to the welfare of households, as the partial (4.18) says. Note also that $J(k,h)$ is increasing in two constants Ω_X and Ω_Y (lower left panel). These constants measure the contribution of physical and human capital to welfare, respectively.

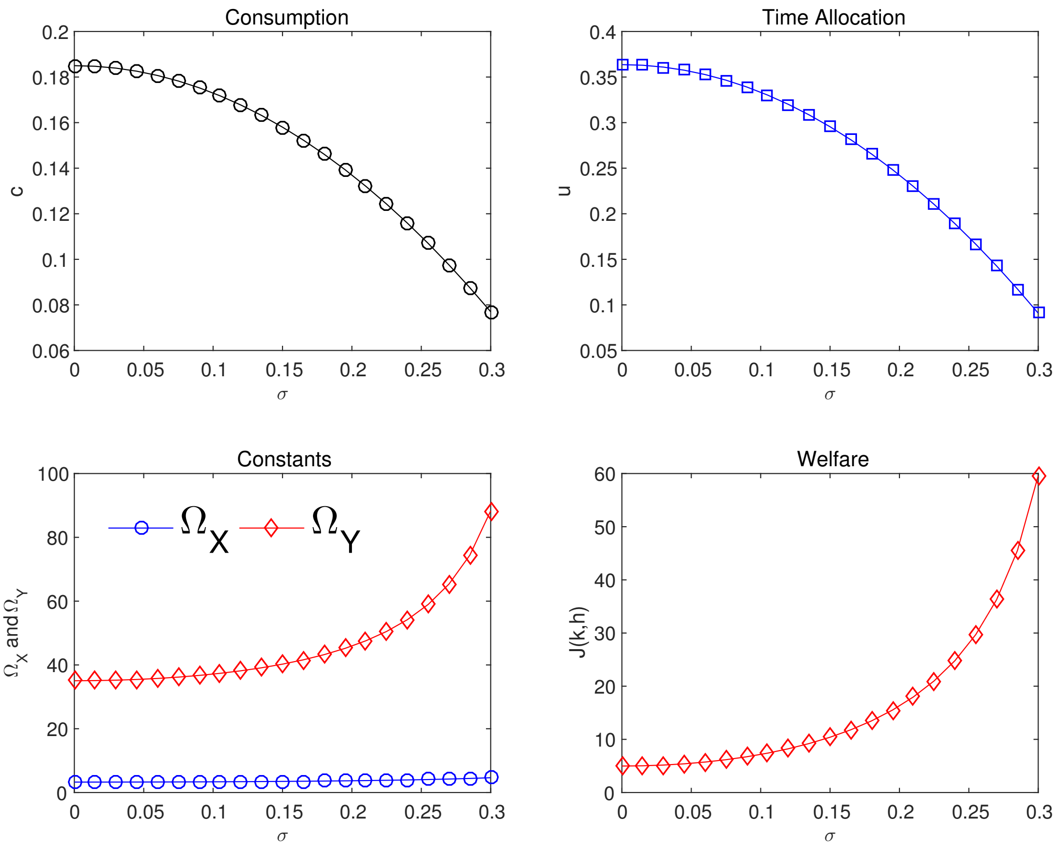

Figure 4.1 Effects of larger demographic uncertainty on per capita consumption c, time allocation u, two constants Ω_X and Ω_Y, and welfare $J(k,h)$.

It may be counterintuitive that higher uncertainty is welfare-improving. Here, however, it leads to the further accumulation of per capita human capital h. And since the welfare $J(k, h)$ is the increasing function of the state variable h, welfare is improved in response to higher uncertainty. Informally, you might think of this as a sudden influx of skilled immigrants. It may increase the stock of human capital in the economy, thereby improving welfare. Therefore, as in Lester et al. (2014), Cho et al. (2015), and Xu (2017), I find that, in the context of demographic uncertainty, there exists a *positive* relationship between welfare and uncertainty. The findings of this section can be summarized as follows:

Proposition 4.1. *With one parameter restriction of Xie (1991), it is possible to find the closed-form solution to the stochastic Uzawa-Lucas model in which population dynamics follows a geometric Brownian motion process. Higher uncertainty unambiguously reduces the optimal level of consumption, while improves the welfare of households.*

So far, I have ignored the role of technological progress. There seems to be, however, a relationship between population growth and technological progress − so-called *scale effects*.[1] It would be interesting to consider them, especially by comparing my findings with those of Bucci et al. (2011), Marsiglio and La Torre (2012a, 2012b), Hiraguchi (2013) and Tsuboi (2018), because they ignore the *link* between population dynamics and technological progress. As such, in the next section, I extend the baseline model of this section by adding technological progress. As we will see, it generates more fruitful insights into the interplay between consumption, uncertainty, and welfare.

4.3 The Model with Technological Progress

In the previous section, we examined the effects of demographic uncertainty on consumption and welfare with no reference to technological progress. In this section, we enrich our analysis by considering the *interaction* between demographic

[1] See Jones (1999; 2005) for an excellent discussion on the scale effect in the context of endogenous growth models.

uncertainty and technology shocks. We will see whether adding technological progress radically alters the implications of the previous section.

4.3.1 Technology and Capital Accumulation

To generalize the above model, I now add technology $A(t)$ to the production function (4.1):

$$Y(t) = (u(t)H(t))^{\alpha} K(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}, \tag{4.19}$$

where, following Bucci et al. (2011) and Hiraguchi (2013), I assume that technological progress is driven by a geometric Brownian motion process:

$$dA(t) = \mu A(t)dt + \sigma_a A(t) dz_a(t), \tag{4.20}$$

where $\mu > 0$ is the rate of technological progress and $\sigma_a \geq 0$ is the diffusion coefficient of technology. Again, as $\sigma_a \to 0$, we would recover the deterministic limit, which would be called "nonstochastic steady state" in the RBC literature. Here, unlike previous studies, we are more interested in the relationship between demographic uncertainty and technology shocks.[1] In consequence, I assume $(dz_L)(dz_a) = \eta dt$, with η being the correlation coefficient between $dz_L(t)$ and $dz_a(t)$. As η is going to play a pivotal role below, it would be instructive to explain what is meant by $\eta > 0$ or $\eta < 0$ at this stage.

First, examples of the positive correlation of population dynamics and technological progress ($\eta > 0$) would include scale effects or brain gain. Let us consider the latter case in more detail. Suppose that, because of natural disasters (such as earthquakes), or political pressures, or war, high-skilled immigrants suddenly come from overseas. Then, as the number of "potential" innovators gets larger in the home country, the possibility of innovation may become strong. In other words, a larger population boosts technological progress. This is the $\eta > 0$ scenario.

Second, in contrast, the $\eta < 0$ scenario is less intuitive than the $\eta > 0$ scenario. But consider the following *pro-labor-saving effects*: suppose that a country has

[1] A *negative* technology shock might be caused by natural disaster, environmental degradation or social disorder. See Aiyar et al. (2008) for examples of technological *regress*.

only two inputs for the production of final goods $Y(t)$; technology $A(t)$ and labor $L(t)$. Then, when a number of population suddenly gets smaller due to war, for instance, a country would be forced to resort to the other input − technology. Put differently, given the unexpected labor shortage, it may start to look for the new way to produce final goods without labor. In that situation, firms would be more willing to increase R&D spending, so that they can promote technological progress and produce a sufficient amount of goods with fewer labors. In sum, the smaller number of population may encourage a country to put more emphasis on the allocation of resources to R&D (and hence technological progress).[1] This is the $\eta < 0$ scenario.

Having understood the meaning of positive or negative correlation η, we can rewrite the associated production function when population dynamics is tied to technology as:

$$y_a(t) = (u(t)h(t))^{\alpha} k(t)^{\beta} A(t)^{1-\alpha-\beta}. \tag{4.21}$$

With these modifications, we can rewrite the stochastic Uzawa-Lucas equation (4.4) as

$$dh(t) = (b(1-u(t)) - (n+\mu+\delta_H - \sigma_a^2 - \sigma_L^2 - \eta\sigma_a\sigma_L))hdt - (\sigma_a dz_a(t) + \sigma_L dz_L(t))\, h, \tag{4.22}$$

and the corresponding stochastic Solow equation (4.5) as

$$\begin{aligned} dk(t) = &\left(\underbrace{(u(t)h(t))^{\alpha} k(t)^{\beta} A(t)^{1-\alpha-\beta}}_{\equiv y_a(t)} - (n+\mu+\delta_K - \sigma_a^2 - \sigma_L^2 - \eta\sigma_a\sigma_L)k(t) - c(t) \right) dt \\ &- (\sigma_a dz_a(t) + \sigma_L dz_L(t))\, k, \end{aligned} \tag{4.23}$$

thanks to Itô's Lemma. Eqs. (4.22) and (4.23) are the generalized version of those in the previous section.

[1]Mathematically, consider the "AL" model $Y(t) = A(t)L(t)$. If $L(t)$ gets small unexpectedly, the only way to sustain $Y(t)$ is to increase $A(t)$. This is the essence of *pro-labor-saving effects*.

4.3.2 Stochastic Optimization

Preferences of a representative household are still given by the CRRA utility function (4.6). So, a representative household maximizes its expected utility (4.6) subject to two stochastic processes (4.22) and (4.23) which are now *correlated*. To solve this more general problem, let $\mathfrak{J}(k,h,A)$ be the new value function. The relevant HJB equation is

$$\max_{\{c,u\}}\left(\frac{c^{1-\phi}-1}{1-\phi}-\rho\mathfrak{J}(k,h,A)+\mathfrak{J}_K\left(y_a-(n+\mu+\delta_K-\sigma_a^2-\sigma_L^2-\eta\sigma_a\sigma_L)k-c\right)\right.$$
$$+\mathfrak{J}_H h\left(b(1-u)-(n+\mu+\delta_H-\sigma_a^2-\sigma_L^2-\eta\sigma_a\sigma_L)\right)+\mathfrak{J}_{KH}kh(\sigma_a^2+\sigma_L^2+2\eta\sigma_a\sigma_L)$$
$$\left.+\frac{\mathfrak{J}_{KK}k^2}{2}(\sigma_a^2+\sigma_L^2+2\eta\sigma_a\sigma_L)+\frac{\mathfrak{J}_{HH}h^2}{2}(\sigma_a^2+\sigma_L^2+2\eta\sigma_a\sigma_L)+\mathfrak{J}_A\mu A+\frac{\mathfrak{J}_{AA}\sigma_a^2}{2}A^2\right), \tag{4.24}$$

where $\mathfrak{J}_K\equiv\partial\mathfrak{J}(k,h,A)/\partial k$, $\mathfrak{J}_H\equiv\partial\mathfrak{J}(k,h,A)/\partial h$, $\mathfrak{J}_A\equiv\partial\mathfrak{J}(k,h,A)/\partial A$, $\mathfrak{J}_{KH}\equiv\partial^2\mathfrak{J}(k,h,A)/\partial k\partial h$, $\mathfrak{J}_{KK}\equiv\partial^2\mathfrak{J}(k,h,A)/\partial k^2$, $\mathfrak{J}_{HH}\equiv\partial^2\mathfrak{J}(k,h,A)/\partial h^2$, and $\mathfrak{J}_{AA}\equiv\partial^2\mathfrak{J}(k,h,A)/\partial A^2$. First-order conditions for c is still (4.8), while that for u is slightly modified:

$$u=\frac{k^{\frac{\beta}{1-\alpha}}}{h}\left(\frac{\alpha\mathfrak{J}_K}{b\mathfrak{J}_H}\right)^{\frac{1}{1-\alpha}}A^{\frac{1-\alpha-\beta}{1-\alpha}}. \tag{4.25}$$

Substituting first-order conditions (4.8) and (4.25) back to the HJB equation (4.24) and rearranging, we get a maximized HJB equation:

$$0=\frac{\phi}{1-\phi}\mathfrak{J}_k^{\frac{\phi-1}{\phi}}-\frac{1}{1-\phi}-\rho\mathfrak{J}(k,h,A)-\mathfrak{J}_k k(n+\mu+\delta_K-\sigma_a^2-\sigma_L^2-\eta\sigma_a\sigma_L)$$
$$-\mathfrak{J}_H h(n+\mu+\delta_H-b-\sigma_a^2-\sigma_L^2-\eta\sigma_a\sigma_L)+\left(\frac{1-\alpha}{\alpha}\right)\alpha^{\frac{1}{1-\alpha}}b^{\frac{\alpha}{\alpha-1}}k^{\frac{\beta}{1-\alpha}}\mathfrak{J}_K^{\frac{1}{1-\alpha}}\mathfrak{J}_H^{\frac{\alpha}{\alpha-1}}$$
$$+(\sigma_a^2+\sigma_L^2+2\eta\sigma_a\sigma_L)\left(\frac{\mathfrak{J}_{KK}k^2}{2}+\frac{\mathfrak{J}_{HH}h^2}{2}+\mathfrak{J}_{KH}kh\right)+\mathfrak{J}_A\mu A+\frac{\mathfrak{J}_{AA}\sigma_a^2}{2}A^2. \tag{4.26}$$

With this equation, using the same technique above, we can solve it in closed form. It can be summarized as follows:

Theorem 4.2. When $\phi = \beta$, there exists the closed-form representation of the value function that satisfies TVC of the form:

$$\mathfrak{J}(k, h, A) = \Theta_X k^{1-\beta} + \Theta_Y h^{\alpha} A^{1-\alpha-\beta} + \Theta_Z, \tag{4.27}$$

where $\Theta_Z = \Omega_Z$,

$$\Theta_X \equiv \frac{1}{1-\beta}\left(\frac{\beta}{\rho + (1-\beta)\left(\mathbb{S}_k - \sigma_a^2 - \sigma_L^2 - \eta\sigma_a\sigma_L + \frac{\beta(\sigma_a^2 + 2\eta\sigma_a\sigma_L + \sigma_L^2)}{2}\right)}\right)^{\beta}, \tag{4.28}$$

and

$$\begin{aligned}\Theta_Y \equiv& \frac{\Theta_X(1-\beta)}{b^{\alpha}}\\ &\times\left(\frac{1-\alpha}{\rho - \alpha\left(b - \mathbb{S}_h + \alpha\eta\sigma_a\sigma_L + \frac{(1-\alpha)(\sigma_a^2+\sigma_L^2)}{2} + (1-\alpha-\beta)\left(\mu + \frac{\sigma_a^2}{2}(\alpha+\beta)\right)\right)}\right)^{1-\alpha}.\end{aligned} \tag{4.29}$$

The explicit expressions for two control variables are:

$$\begin{aligned} c =& \Theta_X^{-\frac{1}{\beta}}(1-\beta)^{-\frac{1}{\beta}}k \\ =& \frac{\rho + (1-\beta)\left(\mathbb{S}_k - \sigma_a^2 - \sigma_L^2 - \eta\sigma_a\sigma_L + \frac{\beta(\sigma_a^2+2\eta\sigma_a\sigma_L+\sigma_L^2)}{2}\right)}{\beta}k,\end{aligned} \tag{4.30}$$

and

$$\begin{aligned} u =& \left(\frac{(1-\beta)\Theta_X}{b\Theta_Y}\right)^{\frac{1}{1-\alpha}} \\ =& \frac{\rho - \alpha\left(b - \mathbb{S}_h + \alpha\eta\sigma_a\sigma_L + \frac{(1-\alpha)(\sigma_a^2+\sigma_L^2)}{2} + (1-\alpha-\beta)\left(\mu + \frac{\sigma_a^2}{2}(\alpha+\beta)\right)\right)}{b(1-\alpha)}.\end{aligned} \tag{4.31}$$

Proof. See Appendix A.[1]

4.3.3 Discussion

As in the previous section, I in turn comment on the implications of two shocks for the optimal level of consumption and welfare.

4.3.3.1 Consumption

It is appropriate first to understand the effect of the higher correlation on the optimal level of (per capita) consumption, as it was absent in the previous section and previous studies. We can see that

$$\frac{\partial c}{\partial \eta} < 0, \tag{4.32}$$

that is, the higher correlation reduces the optimal level of (per capita) consumption. Since this was missing in the previous studies, it is worth thinking about why. Remember that, in the previous section, we found that the larger shock would reduce the optimal level of consumption, due to a precautionary saving motive. Here, because two shocks are correlated, the influence of one shock is *amplified* by the other. For example, think about the case of positive correlation, $\eta > 0$. In that case, the technology shock *boosts* demographic uncertainty, and vice versa. As such, the higher correlation generates higher uncertainty, hence a stronger precautionary saving motive and less per capita consumption.

Having comprehended the implications of the higher correlation for per capita consumption, let us revisit the relationship between demographic shocks and per capita consumption. One can show that

$$\frac{\partial c}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \lesseqgtr -\frac{(1-\beta)\eta}{2-\beta}\sigma_a, \tag{4.33}$$

in other words, as opposed to the previous section, *when demographic shocks are tied to technology shocks, the effect of demographic uncertainty on the optimal level of consumption is indeterminate.* This is illustrated in Fig. 4.2. The line with circles ($\eta = 0$) is the benchmark; as we found above, higher demographic

[1]The condition for $u \in (0, 1)$ is lengthy and can be obtained by straightforward calculation.

uncertainty reduces consumption. When two stochastic processes are positively correlated (the line with squares, $\eta = 1$), as the partial (4.32) implies, consumption is further reduced. The reason is that, in this case, when the size of one shock gets larger, that of the other also gets larger, resulting in a much stronger precautionary saving motive. Thus, agents consume *less* and save *more* for the more uncertain future.

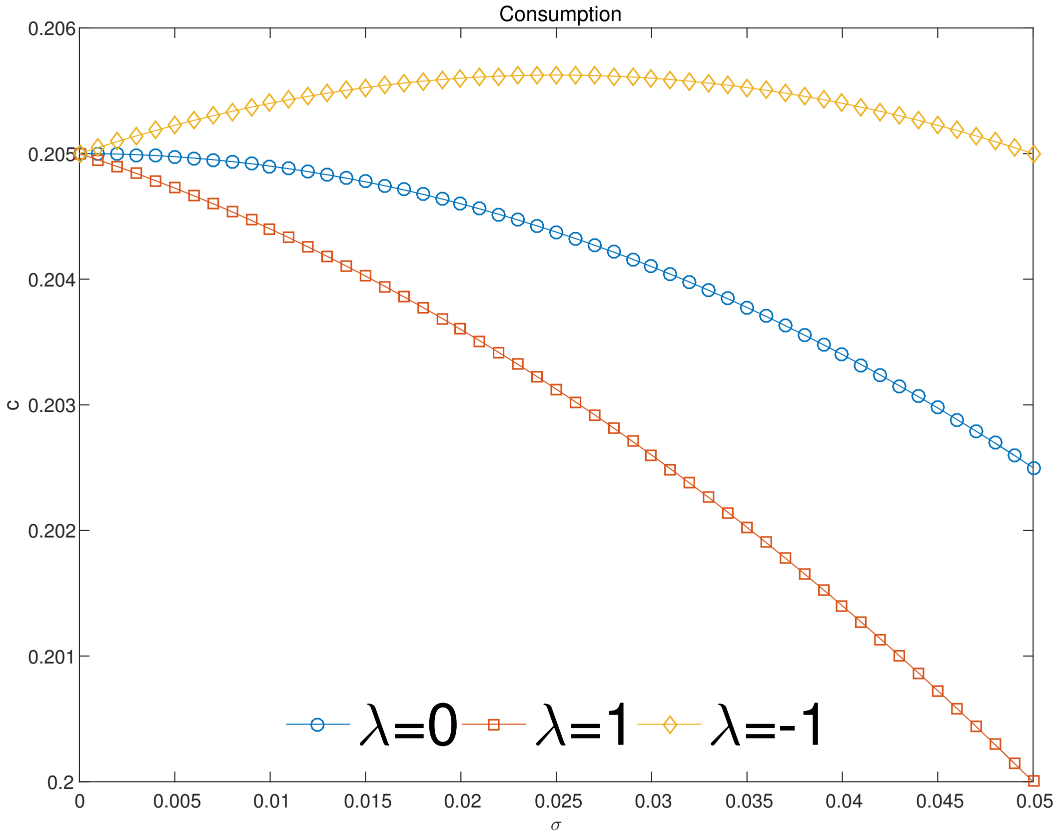

Figure 4.2 Consumption per capita and demographic uncertainty with correlation.

The interesting case would be when two processes are *negatively* correlated (the line with diamonds, $\eta = -1$). We can see an *inverted U-shaped* association between the optimal level of per capita consumption and demographic uncertainty: it initially *rises*, and then falls. Intuitively, when two stochastic processes are negatively correlated, one process "quiets down" the other process. It mitigates the overall effect of uncertainty on a precautionary saving motive, as long as uncertainty is moderate. This yields the *positive* relationship between c and σ_L. Put differently, agents consume *more* and save *less* as long as one shock weakens the other. When the shock is large, however, the usual force outweighs this effect. On that account, consumption starts to fall in response to higher uncertainty. This is the mechanism through which there emerges an inverted U-shaped relation between the optimal level of per capita consumption and demographic uncertainty, when two stochastic processes are negatively correlated.

What about the technology shock? In fact, we have

$$\frac{\partial c}{\partial \sigma_a} \gtreqless 0 \Leftrightarrow \sigma_a \lesseqgtr -\frac{(1-\beta)\eta}{2-\beta}\sigma_L,$$

namely, the influence of technology shocks on per capita consumption is also *ambiguous*. As illustrated in Fig. 4.3, it is governed by the correlation parameter η. As the underlying mechanism is the same with that for demographic uncertainty, I don't repeat the same explanation.

Taking stock, I analytically characterize the relationship between consumption and demographic/technology shocks. It is crucially different from Bucci et al. (2011) and Marsiglio and La Torre (2012a) – who argue that larger shocks *always* reduce consumption – and Marsiglio and La Torre (2012b) and Hiraguchi (2013) – who find that shocks have *nothing* to do with consumption. Instead, when population dynamics interacts with technological progress, the impacts of higher uncertainty on the optimal level of (per capita) consumption are *indeterminate*. The implications for the design of optimal policy are that demographic policies should not be implemented with no reference to the state of technology.

4.3.3.2 Welfare

For our purpose, it would be instructive to begin with the partial

$$\frac{\partial \mathcal{J}(k,h,A)}{\partial \eta} > 0, \tag{4.34}$$

so, the higher correlation is welfare-improving. To see why, note that

$$\frac{\partial u}{\partial \eta} < 0.$$

This says that the higher correlation causes people to accumulate further human capital. As the closed-form representation of the value function (4.27) shows, wel-

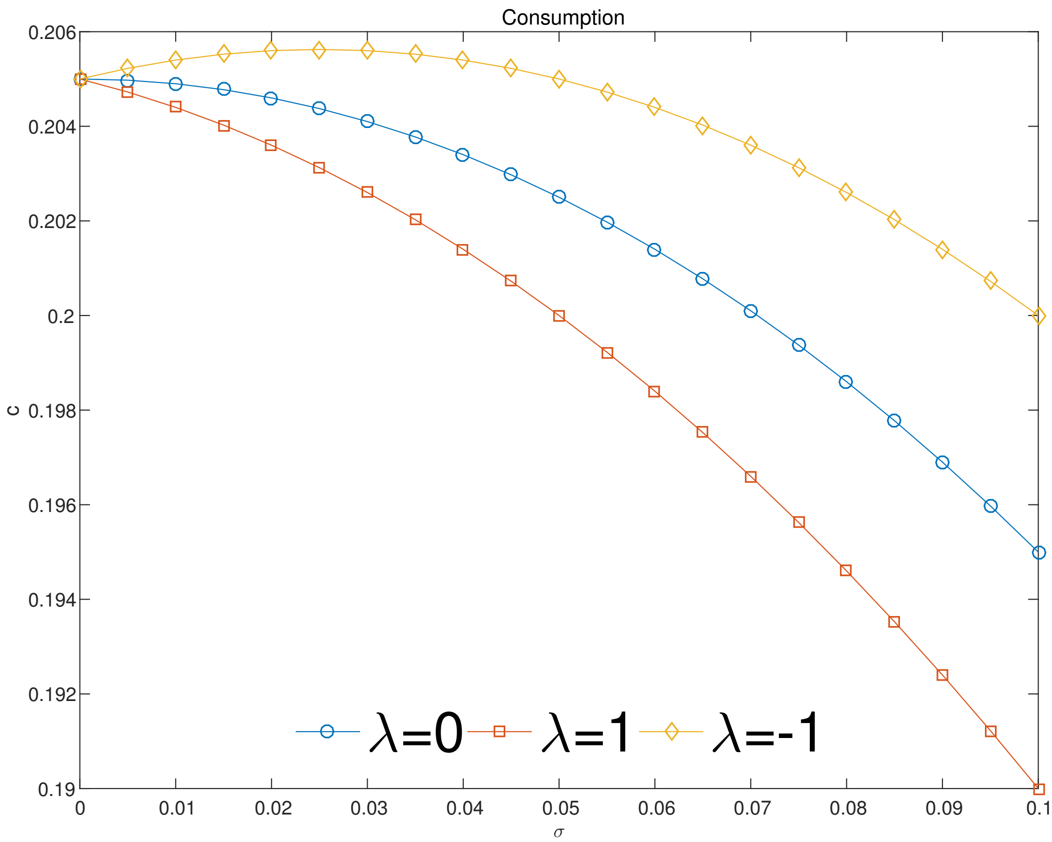

Figure 4.3 Consumption per capita and technology shocks.

fare is improved by the accumulation of human capital. Therefore, the higher correlation, through human capital accumulation, improves welfare. As such, the higher correlation between population dynamics and technological progress is welfare-improving.

With this in mind, let us next examine what impact demographic and technology shocks have on welfare. In the absence of technology, as we saw above, demographic shocks are always welfare-improving, as they lead to human capital accumulation. When technology is tied to demographic shocks, we have

$$\frac{\partial u}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \lesseqgtr -\frac{\alpha\eta}{1+\alpha}\sigma_a. \tag{4.35}$$

Thus, the sign is *indeterminate*; the effect of demographic uncertainty on u is now ambiguous. This is illustrated in Fig. 4.4. The left panel shows the relationship between σ_L and u, while the right panel shows that between σ_L and welfare. In the benchmark case of $\eta = 0$ (the line with circles), the larger demographic shock improves welfare, as we saw in the previous section. When $\eta > 0$ (the line with squares), welfare is further improved in accordance with the larger shock, for the reason discussed in the previous section.

The interesting case is again when two stochastic processes are negatively correlated ($\eta < 0$, the line with diamonds). Welfare is initially *reduced* by higher demographic uncertainty, but when they are beyond the threshold value $\sigma_L = -\alpha\eta\sigma_a/(1+\alpha)$, welfare starts to ameliorate; so there emerges the *U-shaped* relationship between welfare and demographic uncertainty. This result is poles apart in the previous studies of Cho et al. (2015), Lester et al. (2014), and Xu (2017). Therefore, it is worth considering what is going on.

When technology is negatively tied to demographic uncertainty, the larger technology shock mitigates the welfare-improving force of demographic uncertainty in the previous section. Thus, as long as demographic uncertainty is small, its beneficial impact is offset by the larger technology shock. At the certain level $\sigma_L = -\alpha\eta\sigma_a/(1+\alpha)$, however, the former overtakes the latter, hence welfare is improved. This channel generates a U-shaped relationship between welfare and demographic shocks when technology is negatively tied to demographic uncertainty.

What about the impact of technology shocks on $\mathcal{J}$? In the same vein, we have

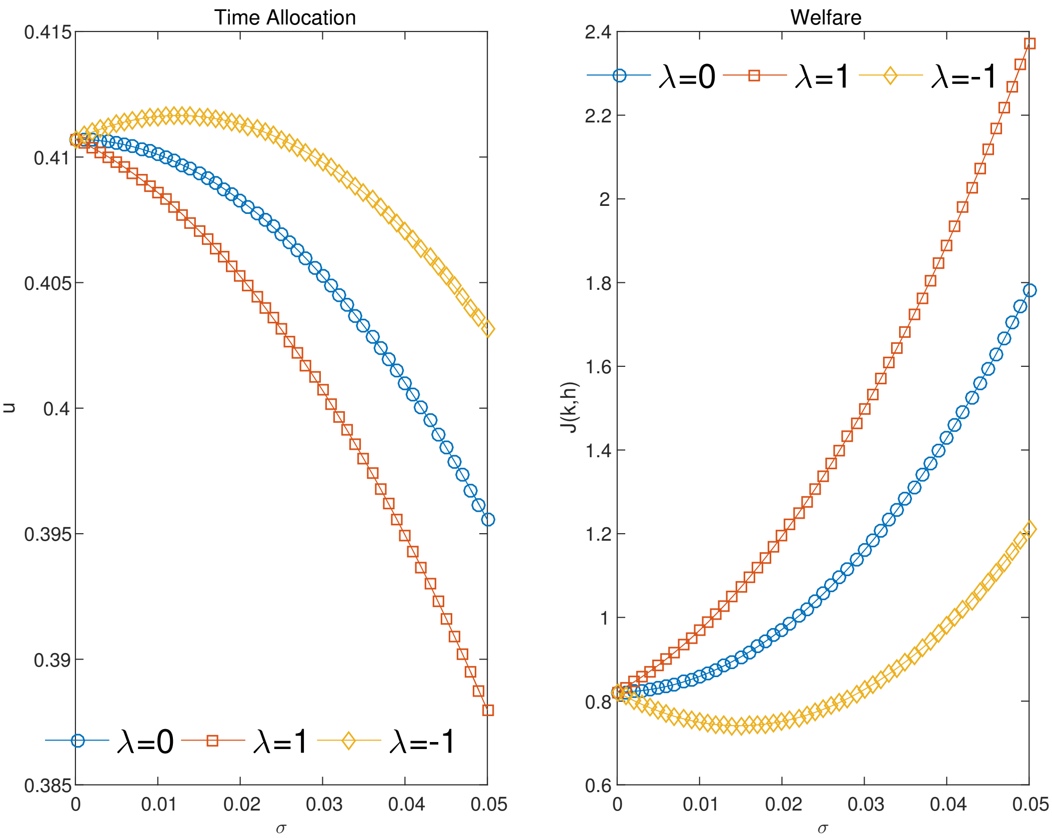

Figure 4.4 The relationship between demographic shocks and time allocation (left), and that between demographic shocks and welfare (right).

$$\frac{\partial u}{\partial \sigma_a} \gtreqless 0 \Leftrightarrow \sigma_a \lesseqgtr -\frac{\alpha\eta}{(1+\alpha)(\alpha+\beta)(1-\alpha-\beta)}\sigma_L,$$

namely, the welfare-implications of technology shocks are again governed by η, as displayed in Fig. 4.5. Overall, the qualitative effects of technology shocks turn out to be quite identical to that of demographic uncertainty. The findings of this section can be summarized as follows:

Proposition 4.2. *The parameter restriction of Xie (1991) enables a closed-form solution to the stochastic Uzawa-Lucas model in which both population dynamics and technological progress follow the correlated Brownian motion process. When technology is tied to demographic uncertainty, the effect of demographic or technology shocks is governed by the correlation coefficient. When two stochastic processes are untied or positively tied to each other, the larger shock reduces per capita consumption, but improves welfare. On the other hand, when they are negatively tied, there emerges an inverted U-shaped relation between uncertainty and consumption, and a U-shaped relation between uncertainty and welfare.*

4.4 Concluding Remarks

Many economic decisions involve uncertainty about the outcome of the choice we make. As we cannot know the final result of the decision until it occurs, it is important to understand how uncertainty affects the optimal level of consumption and the welfare of households. The standard narrative would be that, in the presence of uncertainty, consumption is reduced via a precautionary saving motive, and subsequently, welfare is deteriorated. Recent several studies, however, find the above intuitive narrative not necessarily true. Some find that higher uncertainty reduces consumption, while others argue that there is no relationship between them. Constructing the stochastic Uzawa-Lucas model in which both population dynamics and technological progress follow the *correlated* Brownian motion process, I address this theoretical controversies. When technology is untied to demographic uncertainty, they always reduce consumption. When they

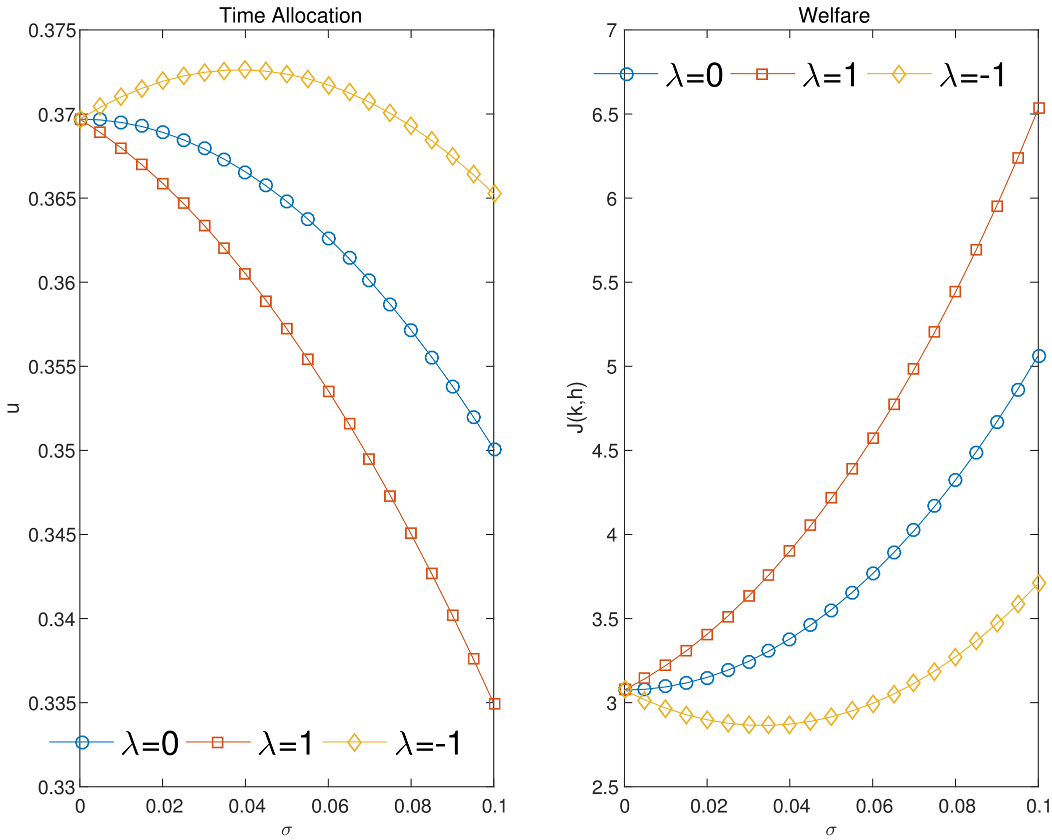

Figure 4.5 The relationship between technology shocks and time allocation (left), and that between technology shocks and welfare (right).

are negatively tied to each other, however, there emerges an inverted U-shaped relationship between the size of shocks and consumption.

I also analyze the impact of shocks on welfare. I find that demographic shocks are always welfare-improving when they are untied to technology. But, when they are negatively tied, there emerges a U-shaped relationship between the size of shocks and the welfare of households. All my findings are completely characterized by the closed-form solution to the stochastic two-sector optimal growth model of Uzawa (1965) and Lucas (1988). As such, this chapter also contributes to the large literature that have tried finding the explicit solution to stochastic growth models recently surveyed by Wälde (2011a; b).

To summarize for the purpose of policy implications, this chapter shows that the effects of demographic uncertainty are ambiguous when they are negatively tied to technology. Therefore, when policymakers wish to implement unprecedented demographic policies, such as the end of one-child policy in China that would generate the considerable amount of uncertainty about the future population dynamics, they have to prudently take the state of technology into account. Otherwise, the policy won't achieve the desirable goal.

4.A Guide to Analytical Solutions

In this Appendix, I briefly explain how to find the functional form of a value function. I use the standard "guess and verify" method to find the closed-form solution. The exposition here is based on Appendix A of Bucci et al. (2011).

I postulate the tentative value function of the form:

$$\mathcal{J}(k, h, A) = \Theta_X k^{\zeta_1} + \Theta_Y h^{\zeta_2} A^{\zeta_3} + \Theta_Z,$$

where Θ_X, Θ_Y, Θ_Z, ζ_1, ζ_2, and ζ_3 are all unknown constants to be determined. The resulting first and second partials with respect to per capita physical capital, human capital, and technology are $\mathcal{J}_K = \zeta_1 \Theta_X k^{\zeta_1 - 1}$, $\mathcal{J}_H = \zeta_2 \Theta_Y h^{\zeta_2 - 1} A^{\zeta_2}$, $\mathcal{J}_A = \zeta_3 \Theta_Y h^{\zeta_2} A^{\zeta_3 - 1}$, $\mathcal{J}_{KK} = \zeta_1(\zeta_1 - 1)\Theta_X k^{\zeta_1 - 2}$, $\mathcal{J}_{HH} = \zeta_2(\zeta_2 - 1)\Theta_Y h^{\zeta_2 - 2} A^{\zeta_3}$, and $\mathcal{J}_{AA} = \zeta_3(\zeta_3 - 1)\Theta_Y h^{\zeta_2} A^{\zeta_3 - 2}$. You can substitute these partials and first-order conditions (4.8) and (4.25) into the HJB equation (4.24). Setting $\zeta_1 = 1 - \phi$, $\zeta_2 = \alpha$,

$\zeta_3 = 1 - \alpha - \beta$, and imposing $\phi = \beta$, and collecting terms, you can obtain the maximized HJB equation. It yields constants in Theorem 4.2. You can get (4.11), (4.12), and (4.13) in Theorem 4.1 by abstracting from stochastic technological progress and correlation.

Moreover, one can establish that the appropriate TVC is satisfied. See Appendix B of Hiraguchi (2013) for an excellent proof of the TVC for stochastic Uzawa-Lucas models in which technological progress is driven by a geometric Brownian motion process. The proof requires the verification theorem. See Chang (2004) for details of this. As a reference, the TVC to be satisfied is $\lim_{t\to\infty} E[e^{-\rho t}k^{1-\beta}] = \lim_{t\to\infty} E[e^{-\rho t}h^{\alpha}A^{1-\alpha-\beta}] = 0$. This is essentially satisfied by the condition for $u \in (0,1)$.

4.B Growth

Though the main text is primarily about welfare, it is indeed also on growth (as in previous chapters). This Appendix collects some mathematical expressions to clarify this point.

The expected growth rate of human capital $\mathcal{G}^H$ is

$$\begin{aligned}\mathcal{G}^H &\equiv \mathbb{E}\left(\frac{\dot{h}(t)}{h(t)}\right)\\ &= \frac{b-\rho-n-\delta_H-(1-\alpha(1-\alpha(1-\alpha-\beta)))\mu+(1-\alpha)\left(1+\frac{\alpha}{2}\right)(\sigma_a^2+\sigma_L^2)}{1-\alpha}\\ &+\frac{(1-\alpha(1-\alpha))\,\eta\sigma_a\sigma_L+\frac{\alpha(\alpha+\beta)(1-\alpha-\beta)}{2}\sigma_a^2}{1-\alpha}.\end{aligned}$$

Differentiating this growth formula, we can find

$$\frac{\partial\mathcal{G}^H}{\partial\eta} = \frac{(1-\alpha(1-\alpha))\sigma_a\sigma_L}{1-\alpha} > 0, \quad \frac{\partial\mathcal{G}^H}{\partial\sigma_L} \gtreqless 0 \Leftrightarrow \sigma_L \gtreqless -\frac{(1-\alpha(1-\alpha)\eta)}{(1-\alpha)(2+\alpha)}\sigma_a,$$

$$\frac{\partial\mathcal{G}^H}{\partial\sigma_a} \gtreqless 0 \Leftrightarrow \sigma_a \gtreqless -\frac{(1-\alpha)(1-\alpha(1-\alpha))\eta}{(1-\alpha)(2+\alpha)+\alpha(\alpha+\beta)(1-\alpha-\beta)}\sigma_L.$$

Thus, a higher correlation unambiguously accelerates growth. In contrast, the effects of two shocks on growth are ambiguous; there exists a threshold value that yields a nonlinear association.

Chapter 5

An Analytical Inquiry into the Growth-Uncertainty Nexus

5.1 Introduction

This final chapter develops the Uzawa-Lucas model that can replicate all four patterns introduced in Ch. 1.[1] For this purpose, I extend Bucci et al. (2011), Hiraguchi (2013), and Tsuboi (2018) by simply relaxing their assumption of no population growth. Moreover, the key feature of my analytical setup is the incorporation of three *correlated* stochastic processes; before directly analyzing how output (or aggregate) uncertainty affects *output* growth, I examine the impacts of human capital uncertainty, physical capital uncertainty, and demographic uncertainty. I then show that this final model can replicate *all* four patterns in Fig. 1.1, depending on how they are correlated. Therefore, this chapter aims to help better understand *why* the results of empirical findings are surprisingly mixed, and put forward a hypothesis to be proved for resolving *empirical ambiguities* in the literature.

Unlike most chapters, I begin with brief literature review. As *empirical* studies are already reviewed in Ch. 1, I focus on *theoretical* studies.

[1]This chapter is based on the substantially revised version of the unpublished paper presented at 13th Macro Conference for Young Economists on February 20, 2019. The older version with Poisson jump process is available at `https://www.jsie.jp/kansai/wp/wp-content/uploads/180519_Tsuboi_Related.pdf`

5.1.1 Theoretical Literature

Compared with empirical studies cited in Ch. 1, *theoretical* studies are scarcer in this literature. Smith (1996) develops a stochastic endogenous growth model based on capital externalities in which output is generated from capital according to the stochastic process. It examines the link between growth and capital externalities, or growth and taxes under uncertainty. Analyzing a monetary growth model, Blackburn and Pelloni (2004) theoretically show that a sign (negative or positive) is dependent on whether uncertainty stems from nominal or real shocks.[1] Femminis (2001) constructs a stochastic endogenous growth model of Uzawa (1965) and Lucas (1988) but doesn't consider human capital uncertainty. Bucci et al. (2011) and Hiraguchi (2013) find closed-form solutions to the stochastic Uzawa-Lucas model in which technological progress is driven by a geometric Brownian motion process. Similarly, Marsiglio and La Torre (2012a,b) find analytical solutions to the stochastic Uzawa-Lucas model with population dynamics following a geometric Brownian motion process. Posch (2011) develops a stochastic Ramsey model with fiscal policy in which technological progress is driven by a geometric Brownian motion process. Combining calibration with panel data estimation, it finds that a sign depends on the type of taxes; for example, it is negative under a labor income tax scheme, while positive under a capital income tax scheme. Posch and Wälde (2011) also stress the importance of taxes in studying the growth-uncertainty nexus in the stochastic growth model with vintage capital whose uncertainty is driven instead by a Poisson jump process.

These (mostly) theoretical studies are similar to this thesis in terms of the model structure. None of them, however, finds a *nonlinear* (U-shaped or inverted U-shaped) relationship. *Nor* none has brought *four links between growth and uncertainty* into its central research question. This chapter fills these two gaps by developing the simple growth model under uncertainty by focusing on new channels highlighted in Fig. 5.1.

This chapter is organized as follows. Sect. 2 sets up the Uzawa-Lucas endogenous growth model featuring the stochastic accumulation of capital and de-

[1] Using micro data, Chong and Gradstein (2009) find that higher uncertainty has either negative or positive effects on growth, and that "the quality of the judiciary" matters in understanding their quantitative relation.

mographic uncertainty. Using key expressions obtained in Sect. 2, I analytically examine the link between growth and uncertainty in Sect. 3. I then briefly analyze how output uncertainty affects output growth in Sect. 4. Concluding remarks appear in Sect. 5.

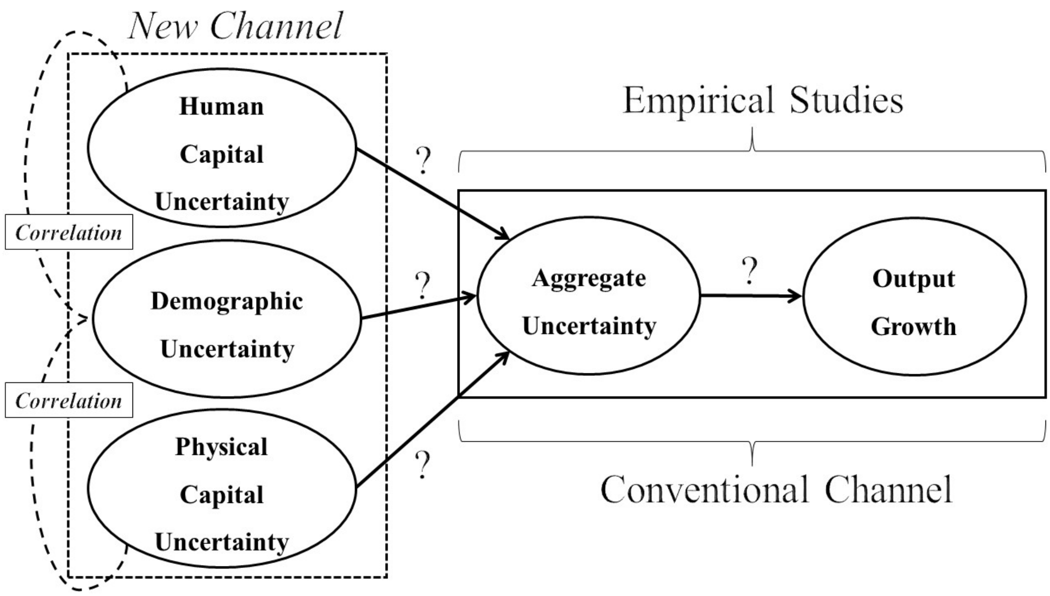

Figure 5.1 New Channels and Model Overview.

5.2 The Model

In this section, I develop the stochastic Uzawa-Lucas model in which the population dynamics and the accumulation of capital follow stochastic processes. Consider a closed economy in continuous time running to an infinite horizon. The economy is inhabited by a large number of identical households. Suppose that the economy admits a representative household, so that the demand and supply side of the economy can be represented as if it resulted from the behavior of a single household.

It is endowed with one unit of time and uses all of that. It either works in a final-goods sector or learns in a human capital sector. There is no other use of time. Part of the human capital in this economy can be used for further human capital accumulation. It captures the technology of an economy to generate human capital, such as school system and on-the-job training (OJT). Let $u(t) \in (0, 1)$

denote the fraction of time spent working to produce final goods $Y(t)$. Thus, the rest $1-u(t)$ represents the fraction of time spent learning. The amount of leisure is exogenously fixed, so there is no choice about it.

5.2.1 Capital Accumulation and Household

The accumulation of human capital $H(t)$ is stochastically governed by

$$dH(t) = b(1-u(t))H(t)dt - \delta_H H(t)dt + \sigma_H H(t)dz_H(t), \tag{5.1}$$

where $b > 0$ is an exogenous parameter that indicates how efficient human capital accumulation is. $\delta_H \in (0,1)$ is its depreciation rate. Net of δ_H and the diffusion term, if no effort is devoted to human capital accumulation ($u(t) = 1$), then none accumulates. If all effort is devoted to this purpose ($u(t) = 0$), $H(t)$ grows at its maximal rate b. In between these extremes, there are no diminishing returns to the stock $H(t)$.

$dz_H(t)$ is the increment of a Brownian motion (or Wiener) process such that the mean $\mathbb{E}(dz_H) = 0$ and variance $\mathcal{V}(dz_H) = dt$. $\sigma_H \geq 0$ is the diffusion coefficient of human capital; if $\sigma_H = 0$, we would recover the deterministic limit. As changes in the process over any finite interval of time are normally distributed, a variance increases linearly with the time interval dt. I assume that the initial stock of human capital $H(0) = H_0 > 0$ is given.

The stochastic process (5.1) is the *controlled* diffusion process; it contains one of key control variables in a Uzawa-Lucas model, $u(t)$, in the drift term. Bucci et al. (2011), Marsiglio and La Torre (2012a,b), and Hiraguchi (2013) all assume that either technological progress or population dynamics is stochastic, while the accumulation of human capital is *deterministic* ($\sigma_H = 0$). This assumption is at odds with the findings of empirical studies on human capital uncertainty, such as Hartog et al. (2007). A lack of human capital uncertainty has also been frequently pointed out by, for instance, Levhari and Weiss (1974) and Krebs (2003). In response to their critique, I assume that the accumulation of human capital is *stochastic*.[1]

[1]Hiraguchi (2018) also considers human capital uncertainty like (5.1) but studies wealth distribution.

Next, the economy-wide resource constraint is

$$dK(t) = \underbrace{(u(t)H(t))^{\alpha}K(t)^{\beta}L(t)^{\gamma}}_{\equiv Y(t)}dt - C(t)dt - \delta_K K(t)dt + \sigma_K K(t)dz_K(t), \quad (5.2)$$

where $K(t)$ is physical capital and $\delta_K \in (0,1)$ is its depreciation rate. $C(t)$ denotes consumption of final goods. $\alpha \in (0,1)$ is the human capital share of income in the generalized Cobb-Douglas production function originally proposed by Mankiw et al. (1992). It is also assumed by Bucci et al. (2011) and Hiraguchi (2013), net of technological progress.[1] $\beta \in (0,1)$ is the physical capital share of income and $\gamma \equiv 1 - \alpha - \beta \in (0,1)$ is the labor share of income. Just like (5.1), $\sigma_K \geq 0$ is the diffusion coefficient of physical capital and dz_K is the associated increment of a Brownian motion process. I assume that the initial stock of physical capital $K(0) = K_0 > 0$ is given as well.

For total population $L(t)$, I relax the assumption of *no* population growth and $L = 1$ in Bucci et al. (2011), Hiraguchi (2013), and Tsuboi (2018). Specifically, following Merton (1975), Chang (1988), Marsiglio and La Torre (2012a; b) and Marsiglio (2014), its law of motion is driven by the stochastic differential equation

$$dL(t) = nL(t)dt + \sigma_L L(t)dz_L(t), \quad (5.3)$$

where n is the growth rate of population. $\sigma_L \geq 0$ is the diffusion coefficient of population growth and $dz_L(t)$ is the associated increment of a Brownian motion process. I again assume that $L(0) = L_0 > 0$, so that $L(t) > 0$ for all t with probability 1. Technically, in the stochastic Ramsey model with (5.3), Smith (2007) finds that the assumption of $L = 1$ shuts the optimal level of consumption off from uncertainty prevailing in an economy. Thus, we need to introduce population growth.

At this point, it is appropriate to start working with per capita versions of

[1] As in Romer (1990, p.S85), the production technology specified in (5.2) implicitly neglects the fact that $H(t)$ and $L(t)$ are supplied jointly. One can imagine that there are some skilled persons who specialize in human capital accumulation and supply no labor. Robertson (2002) develops the deterministic Uzawa-Lucas model with population growth.

(5.1) and (5.2). Using Itô's lemma, the former is

$$dh(t) = b(1-u(t))h(t)dt-(n+\delta_H-\sigma_L^2+\eta_{HL}\sigma_H\sigma_L)h(t)dt+\sigma_H h(t)dz_H(t)-\sigma_L h(t)dz_L(t), \tag{5.4}$$

and the latter is

$$\begin{aligned} dk(t) = {} & \underbrace{(u(t)h(t))^{\alpha}k(t)^{\beta}}_{\equiv y(t)}dt - c(t)dt - (n+\delta_K-\sigma_L^2+\eta_{KL}\sigma_K\sigma_L)k(t)dt \\ & + \sigma_K k(t)dz_K(t) - \sigma_L k(t)dz_L(t), \end{aligned} \tag{5.5}$$

where $h(t) \equiv H(t)/L(t)$ is human capital per capita, $k(t) \equiv K(t)/L(t)$ is physical capital per capita, $y(t) \equiv Y(t)/L(t)$ is output per capita, and $c(t) \equiv C(t)/L(t)$ is consumption per capita. Here, I assume that stochastic processes are *correlated*; that is, $\eta_{HL}dt = (dz_H)(dz_L)$ and $\eta_{KL}dt = (dz_K)(dz_L)$ with η_{HL} and η_{KL} denoting correlation coefficients. We will see that they play a central role when we conduct substantial comparative statics below.

Finally, preferences of a representative household are given by the constant relative risk aversion (CRRA) utility function:

$$\mathbb{E}\int_0^{\infty} e^{-\tilde{\rho}t}\frac{c(t)^{1-\phi}-1}{1-\phi}dt, \tag{5.6}$$

where $\tilde{\rho} \equiv \rho - n$. $\rho > 0$ is the subjective discount rate; the rate at which utility is discounted. $\phi > 0$ is the coefficient of relative risk aversion. As $\phi \to \infty$, households become infinitely risk-averse and infinitely unwilling to substitute consumption over time. $\mathbb{E}$ is the mathematical expectation operator with respect to the information set available to a representative household. Summing up, a representative household maximizes its expected utility (5.6) subject to two stochastic processes (5.4) and (5.5).

5.2.2 Stochastic Optimization

Let $J(k,h)$ denote a value function. The Hamilton-Jacobi-Bellman (HJB) equation is

$$
\begin{aligned}
\tilde{\rho} J(k,h) = \max_{c(t),u(t)} & \frac{c^{1-\phi}}{1-\phi} - \frac{1}{1-\phi} + J_K (uh)^{\alpha} k^{\beta} - cJ_K - J_K(n + \delta_K - \sigma_L^2 + \eta_{KL}\sigma_K\sigma_L)k \\
& + J_H b(1-u)h - J_H(n + \delta_H - \sigma_L^2 + \eta_{HL}\sigma_H\sigma_L)h + \frac{J_{KK}}{2}(\sigma_K^2 - 2\eta_{KL}\sigma_K\sigma_L + \sigma_L^2)k^2 \\
& + \frac{J_{HH}}{2}(\sigma_H^2 - 2\eta_{HL}\sigma_H\sigma_L + \sigma_L^2)h^2,
\end{aligned}
$$

where $J_X \equiv \partial J/\partial X$ and $J_{XX} \equiv \partial J^2/\partial X^2$ for a variable $X = K, H$. First-order conditions are

$$
c = J_K^{-\frac{1}{\phi}}, \tag{5.7}
$$

$$
u = \left(\frac{\alpha J_K}{bJ_H}\right)^{\frac{1}{1-\alpha}} \frac{k^{\frac{\beta}{1-\alpha}}}{h}. \tag{5.8}
$$

Substituting (5.7) and (5.8) into the above HJB equation, after some algebra, we get the *maximized HJB equation*:

$$
\begin{aligned}
0 = & -\tilde{\rho} J(k,h) + \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} k^{\frac{\beta}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} \\
& - J_K(n + \delta_K - \sigma_L^2 + \eta_{KL}\sigma_K\sigma_L)k + J_H(b - \delta_H - n + \sigma_L^2 - \eta_{HL}\sigma_H\sigma_L)h \\
& + \frac{J_{KK}}{2}(\sigma_K^2 - 2\eta_{KL}\sigma_K\sigma_L + \sigma_L^2)k^2 + \frac{J_{HH}}{2}(\sigma_H^2 - 2\eta_{HL}\sigma_H\sigma_L + \sigma_L^2)h^2.
\end{aligned} \tag{5.9}
$$

With this partial differential equation, our task is to *guess and verify* the closed-form representation of the value function $J(k,h)$. As the dynamics of a Uzawa-Lucas model is somewhat complex (even in a *deterministic* context), we must take a cautious approach in getting a closed-form solution. At the steady state, when $\phi \neq \beta$, $u(t)$ is on the transitional path toward its steady state value u^{SS}. In this case, the model doesn't admit an explicit solution. On the other hand, when $\phi = \beta$, we always have $u(t) = u^{SS}$. In the latter case, the model can be solved in closed form. We thus focus on the $\phi = \beta$ case in what follows. So, we have the following theorem:

Theorem 5.1. *When $\phi = \beta$, we can find the closed-form representation of the value function (that satisfies both the HJB equation and the transversality condi-*

tion, or TVC) of the form

$$J(k,h) = \mathbb{X}k^{1-\beta} + \mathbb{Y}h^{\alpha} + \mathbb{Z}, \tag{5.10}$$

where

$$\mathbb{X} \equiv \frac{1}{1-\beta}\left(\frac{\beta}{\tilde{\rho} + (1-\beta)\left(n + \delta_K - \sigma_L^2 + (1-\beta)\eta_{KL}\sigma_K\sigma_L + \frac{\beta(\sigma_K^2+\sigma_L^2)}{2}\right)}\right)^{\beta}, \tag{5.11}$$

$$\mathbb{Y} \equiv \frac{(1-\beta)\mathbb{X}}{b^{\alpha}}\left(\frac{1-\alpha}{\tilde{\rho} - \alpha\left(b - (n + \delta_H - \sigma_L^2) - \alpha\eta_{HL}\sigma_H\sigma_L - \frac{(1-\alpha)(\sigma_H^2+\sigma_L^2)}{2}\right)}\right)^{1-\alpha}, \tag{5.12}$$

$$\mathbb{Z} \equiv -\frac{1}{\tilde{\rho}(1-\beta)}. \tag{5.13}$$

The explicit expressions for control variables are, from Eqs. (5.7) and (5.8),

$$c = \frac{\tilde{\rho} + (1-\beta)\left(n + \delta_K - \sigma_L^2 + (1-\beta)\eta_{KL}\sigma_K\sigma_L + \frac{\beta(\sigma_K^2+\sigma_L^2)}{2}\right)}{\beta}k, \tag{5.14}$$

$$u = \frac{\tilde{\rho} - \alpha\left(b - (n + \delta_H - \sigma_L^2) - \alpha\eta_{HL}\sigma_H\sigma_L - \frac{(1-\alpha)(\sigma_H^2+\sigma_L^2)}{2}\right)}{b(1-\alpha)}. \tag{5.15}$$

Moreover, from (5.4), we can derive the growth formula for human capital:

$$\mathcal{G}_H \equiv \mathbb{E}\left(\frac{\dot{h}}{h}\right) = \frac{b - \tilde{\rho} - n - \delta_H - (1-\alpha(1-\alpha))\eta_{HL}\sigma_H\sigma_L - \frac{\alpha(1-\alpha)}{2}\sigma_H^2 + \frac{\alpha(1-\alpha)+2}{2}\sigma_L^2}{1-\alpha}, \tag{5.16}$$

and, from (5.5), that for physical capital:

$$\mathcal{G}_K \equiv \mathbb{E}\left(\frac{\dot{k}}{k}\right) = \left(\frac{\tilde{\rho} - \alpha\left(b - (n + \delta_H - \sigma_L^2) - \alpha\eta_{HL}\sigma_H\sigma_L - \frac{(1-\alpha)(\sigma_H^2+\sigma_L^2)}{2}\right)}{b(1-\alpha)}\right)^{\alpha} \frac{h^\alpha}{k^{1-\beta}}$$
$$- \frac{\tilde{\rho} + n + \delta_K + \eta_{KL}\sigma_K\sigma_L\left(1 - \beta(1-\beta)\right)}{\beta}$$
$$+ \frac{\sigma_L^2(2 - \beta(1-\beta)) - \sigma_K^2\beta(1-\beta)}{2\beta}, \tag{5.17}$$

where $\dot{h} \equiv dh/dt$ and $\dot{k} \equiv dk/dt$.

Proof. See Appendix 5.A[1]

In Theorem 5.1, we have key closed-form expressions: a value function (5.10), two control variables (5.14) and (5.15), and two types of expected growth rate (5.16) and (5.17). I use these to study the growth-uncertainty nexus in the next section. Before carrying on, however, there is one limitation that deserves some mention: a parameter restriction $\phi = \beta$. Because of the stochastic nature of the model, I must impose it to obtain key expressions in closed form. This restriction, first proposed by Xie (1991) to solve a deterministic endogenous growth model, asserts that the risk aversion parameter equals the physical capital share of income. Consequently, I cannot investigate the case where $\phi \geq 1$ (see Smith (1996) on this point) and the quantitative generality of findings is confined to the *neighborhood* of $\phi = \beta$.

Nevertheless, on this point, Xie (1991, p.430) eloquently puts as follows (notation adapted): "There is, however, no analytical free lunch. To get the explicit dynamics, I have to impose a restriction, $\beta = \phi$, across the preferences and technology. The explicit solution I derive will not generalize to the case in which the parameters β and ϕ differ, but the qualitative results will. Since explicit solutions are the basis for much of our intuition, it is useful to have a new class of models

[1]The condition for $u \in (0, 1)$ is complicated and can be easily computed by straightforward calculation.

that permit them, even if only for a restrictive special case."[1]

Moreover, this parameter restriction has played a key role in the literature on growth with or without uncertainty; Xie (1991, 1994), Rebelo and Xie (1999), Wälde (2005), Smith (2007), Chilarescu (2008), Ruiz-Tamarit (2008), Bucci et al. (2011), Posch (2009; 2011), Posch and Wälde (2011), Marsiglio and La Torre (2012a;b), Hiraguchi (2013; 2014), Tsuboi (2018), and Menoncin and Nembrini (2018) all use this restriction to inspect an underlying mechanism in the most transparent way. I follow their approach: see Wälde (2011) for an extensive discussion on this methodology.

5.3 Comparative Statics

Using key expressions in Theorem 5.1, this section conducts substantial comparative statics. Though we have their closed-form representations, with two correlation parameters and three types of uncertainty, it isn't instructive to list all relevant partial derivatives and carry on with the formal analytics only. Thus, to facilitate the intuition, I supplement an analytical analysis with heavy use of diagrams. In some cases, I simply refer a reader to a self-explanatory diagram, so that you won't see the same point made over and over again.

We begin with human capital uncertainty σ_H. We then analyze physical capital uncertainty σ_K and demographic uncertainty σ_L.

5.3.1 Human Capital Uncertainty

The most important control variable in the Uzawa-Lucas model is u, time spent working. Therefore, we first look at the impacts of σ_H on u, in the absence of correlation $\eta_{HL} = \eta_{KL} = 0$ (our benchmark). From Eq. (5.15), we have

$$\frac{\partial u}{\partial \sigma_H} > 0,$$

[1] See Wälde (2005, p.878) and Posch and Wälde (2011, p.292) for a discussion on the plausibility of this parameter restriction.

as shown in Panel (a) of Fig. 5.2.[1] Higher human capital uncertainty *discourages* people to accumulate human capital in a human capital sector (or *encourages* people to work in a final-goods sector). From Eqs. (5.16) and (5.17), we see

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_H} < 0,$$

because of a *decrease* in time spent learning (panel (c)), and

$$\frac{\partial \mathcal{G}_K}{\partial \sigma_H} > 0,$$

due to an *increase* in time spent working (panel (d)). You may wonder whether uncertainty terms σ_H, σ_K, and σ_L precisely correspond to the *standard deviation* of growth (remember that vertical axis of figures in Ch. 1 measures the standard deviation of growth). In Appendix 5.B, I prove their correspondence in detail. In the present case, human capital uncertainty σ_H and the standard deviation of its expected growth rate σ_g^H are *positively* correlated, as shown in panel (b). So, there is a *negative* relationship between $\mathcal{G}_H$ and its standard deviation σ_g^H as displayed in panel (g), and a *positive* relationship between $\mathcal{G}_K$ and the standard deviation σ_g^H as displayed in panel (h).

For welfare $J(k,h)$, we have

$$\frac{\partial J(k,h)}{\partial \sigma_H} < 0,$$

as shown in panel (e). Why does higher human capital uncertainty deteriorate welfare? Three forces are operating here; first, as $\partial(1-u)/\partial \sigma_H < 0$, in response to higher human capital uncertainty, the expected growth rate of human capital $\mathcal{G}_H$ decreases. As welfare $J(k,h)$ is increasing in h, this human capital contraction is detrimental to welfare. Second, we have $\partial \mathbb{Y}/\partial \sigma_H < 0$, as shown in panel (f). From a value function (5.10), we know that a decrease in $\mathbb{Y}$ means a smaller contribution of h to welfare $J(k,h)$. Third, as $\partial u/\partial \sigma_H > 0$, $\mathcal{G}_K$ increases. This means a larger contribution of k to welfare $J(k,h)$. In total, the sum of the former

[1] Following Mankiw et al. (1992), I choose $\alpha = 1/3$ and $\delta_K = \delta_H = 0.03$. $\beta = 0.36$ is from Ahn et al. (2018). $n = 0.01$ is roughly a world population growth rate. $\rho = 0.05$ is standard in the literature. $b = 0.11$ is from Barro and Sala-i-Martin (2004). Finally, for illustration, I use $\sigma_H = \sigma_K = \sigma_L = 0.010$.

two effects outweighs the final, leading to a net welfare *loss*. Here, we can see that, as Barlevy (2004) shows, welfare is essentially determined by the long-run growth paths chosen by an economy that are altered by changes in human capital uncertainty σ_H.

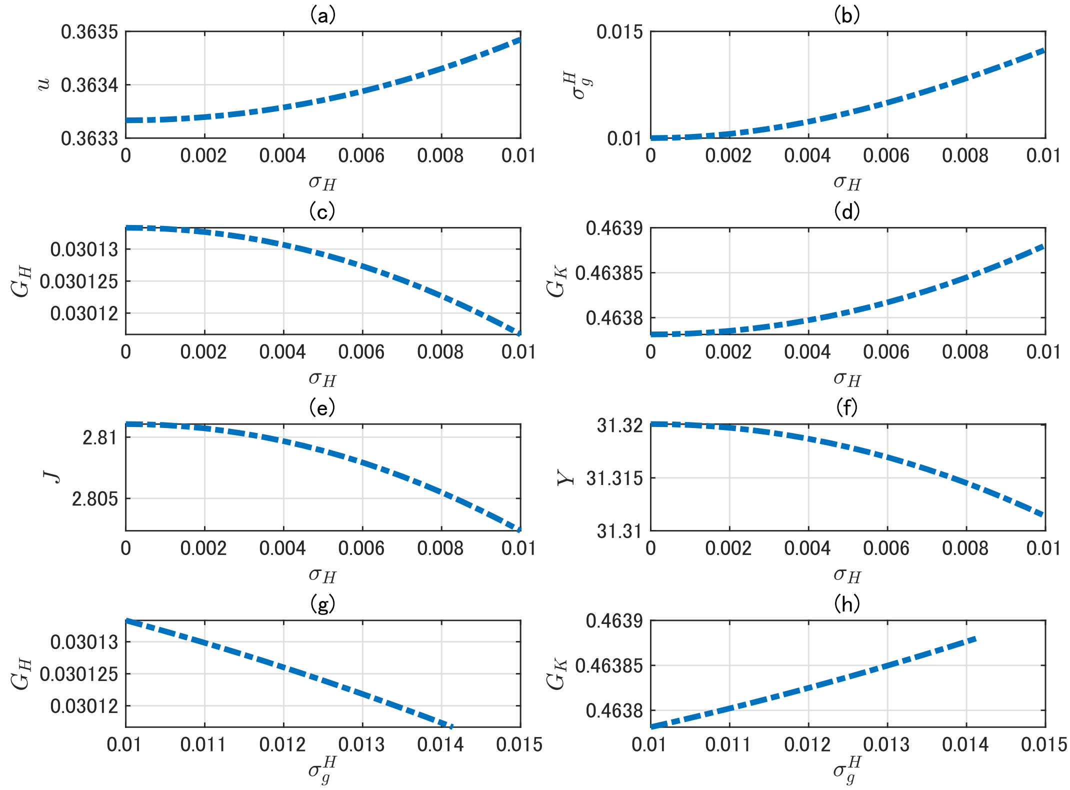

Figure 5.2 The impacts of human capital uncertainty σ_H when $\eta_{HL} = \eta_{KL} = 0$.

What if $\eta_{HL} > 0$ and $\eta_{KL} = 0$? As Fig. 5.3 displays, the qualitative results remain unchanged – except for panels (b), (g), and (h).[1] Due to positive correlation, we have (see Appendix 5.B)

$$\frac{\partial \sigma_g^H}{\partial \sigma_H} \gtrless 0 \Leftrightarrow \sigma_H \gtrless \eta_{HL}\sigma_L.$$

Accordingly, there is a U-shaped association between standard deviation σ_g^H and uncertainty σ_H. Denoting the growth rate that *minimizes* the standard deviation by ^, we have

$$\left.\frac{\partial \mathfrak{G}_H}{\partial \sigma_g^H}\right|_{\mathfrak{G}_H \gtrless \hat{\mathfrak{G}}_H} \gtrless 0, \quad \left.\frac{\partial \mathfrak{G}_K}{\partial \sigma_g^H}\right|_{\mathfrak{G}_K \gtrless \hat{\mathfrak{G}}_K} \gtrless 0.$$

[1]When we consider a correlated case, I will use $\eta_{iL} = 1$ ($i = H, K$) for positive correlation, and $\eta_{iL} = -1$ for negative correlation.

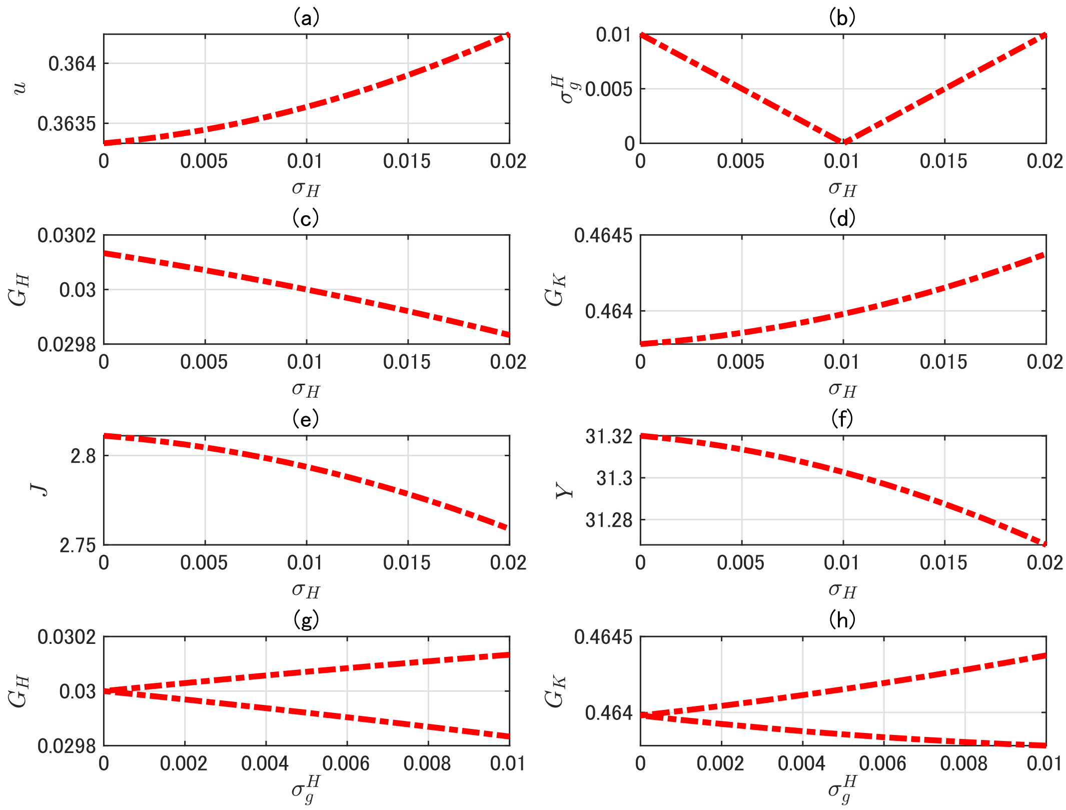

Figure 5.3 The impacts of human capital uncertainty σ_H when $\eta_{HL} > 0$ and $\eta_{KL} = 0$.

These inequalities imply that, when stochastic processes for the accumulation of human capital and population dynamics are *positively* correlated, there exists both positive and negative relationships between growth and uncertainty (panels (g) and (h)). As panels (c) and (d) in Figs. 5.2 and 5.3 are almost identical, this finding is rather due to a statistical property as shown in panel (b).

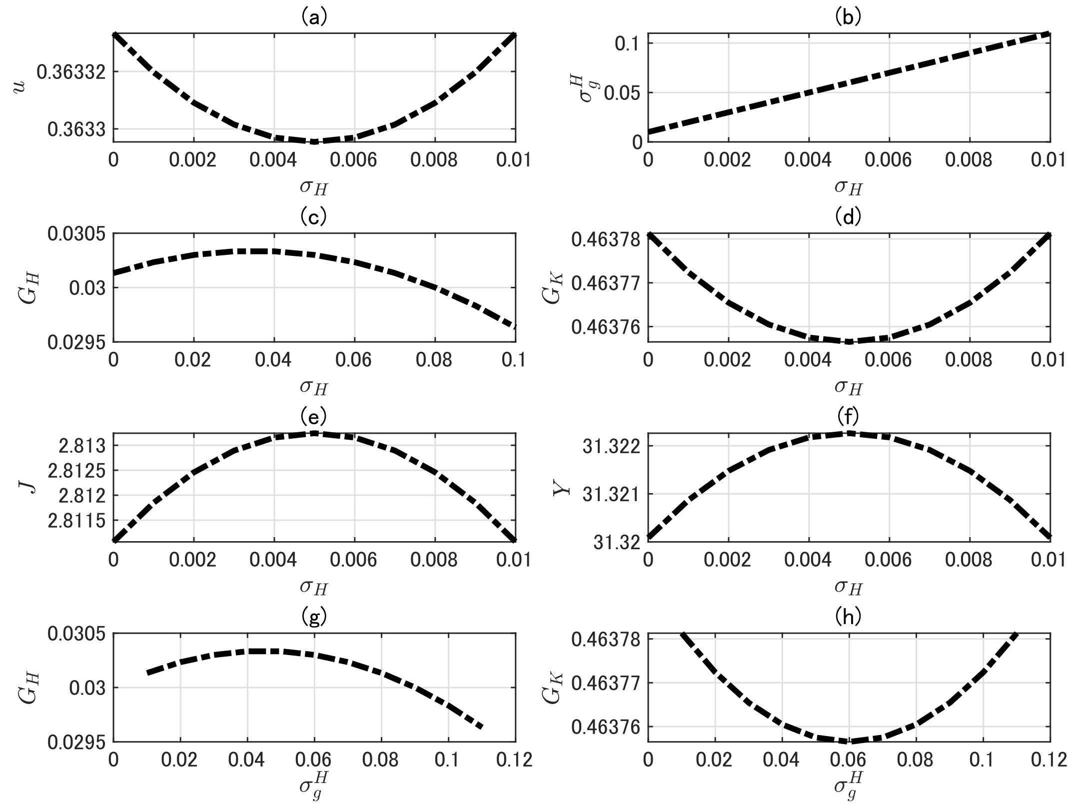

Figure 5.4 The impacts of human capital uncertainty σ_H when $\eta_{HL} < 0$ and $\eta_{KL} = 0$.

Finally, What if $\eta_{HL} < 0$ and $\eta_{KL} = 0$? As Fig. 5.4 displays, we see a large variety of *nonlinear* patterns that have never been found by previous *theoretical* studies. To see the underlying mechanism, let us first look at the impacts of σ_H on u. From Eq. (5.15), we have

$$\frac{\partial u}{\partial \sigma_H} \gtrless 0 \Leftrightarrow \sigma_H \gtrless -\frac{\alpha \eta_{HL}}{1-\alpha}\sigma_L,$$

as shown in Panel (a) of Fig. 5.4. Higher human capital uncertainty, as long as it is moderate, initially encourages people to learn in a human capital sector. When uncertainty exceeds its threshold value $-(\alpha\eta_{HL}/(1-\alpha))\sigma_L$, however, people start to spend more of their time in working, generating a U-shaped dynamics for u.

The key to understanding this unorthodox outcome is

$$\frac{\partial u}{\partial \eta_{HL}} > 0,$$

that is, a higher correlation between the stochastic accumulation of human capital and demographic dynamics encourages people to work. To see why, consider a country with two regions: region A with many schools and region B with few schools. Then, imagine what will happen after an unexpected increase in population. Region A will have more skilled workers than region B. As skills and wage are positively correlated in general, people in region A are more likely to spend their time in working than people in region B, because of their comparative advantage in skills. This is a possible intuition for why a higher correlation encourages people to work longer.

With this in mind, recall that we are thinking of $\eta_{HL} < 0$. This case – probably most realistic case due to an inverse correlation between fertility and schooling levels across countries – can be considered similarly; now imagine a region with only one school but with many children. After a sudden increase in population, more children will be educated, resulting in an initial increase in human capital in a region. With one school only, however, it eventually can't accommodate all children. Knowing not to be able to attend classes, they instead start to spend their time in working in a final-goods sector. It is this *switch* between two sectors that generates a U-shaped dynamics for time allocation u.[1]

Once we understand the dynamics for u, the rest of mechanisms overlaps with those discussed in the absence of correlation; so I won't repeat them. So, we can immediately evaluate the impacts of higher human capital uncertainty on two growth rates:

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_H} \gtrless 0 \Leftrightarrow \sigma_H \lessgtr -\frac{(1-\alpha(1-\alpha))\eta_{HL}}{\alpha(1-\alpha)}\sigma_L, \quad \frac{\partial \mathcal{G}_K}{\partial \sigma_H} \gtrless 0 \Leftrightarrow \sigma_H \lessgtr -\frac{\alpha\eta_{HL}}{1-\alpha}\sigma_L,$$

both of which are displayed in panels (c) and (d), respectively. As σ_H and σ_g^H are positively correlated (panel (b)), we have an *inverted U-shaped* association be-

[1] Another example may include immigration; that is, brain drain and brain gain.

tween growth and uncertainty (panel (g)) proved by García-Herrero and Vilarrubia (2007), and a *U-shaped* relationship (panel (h)) proved by Alimi (2016).

For welfare $J(k,h)$, we have

$$\frac{\partial J(k,h)}{\partial \sigma_H} \gtrless 0 \Leftrightarrow \sigma_H \lessgtr -\frac{\alpha \eta_{HL}}{1-\alpha}\sigma_L,$$

as shown in panel (e). In addition to *three* forces discussed above, note that there is an initial increase in $1-u$, as long as uncertainty is moderate. These *four* conflicting forces are a source of an inverted-U relationship between welfare and human capital uncertainty σ_H. It is also interesting to observe that the level of σ_H maximizing welfare $J(k,h)$ differs from that of σ_H maximizing $\mathcal{G}_H$.

The findings of this subsection can be summarized as follows:

Proposition 5.1. *When there is a positive or no correlation between population dynamics and human capital accumulation, the model predicts a positive or negative relationship between growth and uncertainty. In contrast, when they are negatively correlated, the model predicts a U-shaped or an inverted U-shaped relationship between growth and uncertainty.*

Taking stock, human capital uncertainty can replicate all *four* patterns suggested by empirical studies. It is summarized in the first row of columns (i) to (iv) of Table 5.1.

Table 5.1: Summary of the Growth-Uncertainty Nexus

σ_i	(i) $\partial\mathcal{G}_H/\partial\sigma_g^H$	(ii) $\partial\mathcal{G}_K/\partial\sigma_g^H$	(iii) $\partial\mathcal{G}_K/\partial\sigma_g^K$	(iv) Total	(v) $\partial\mathcal{G}_Y/\partial\sigma_g^Y$
σ_H	$\{+,-,\cap\}$	$\{+,-,\cup\}$	$\{\emptyset\}$	$\{+,-,\cup,\cap\}$	$\{+,-,\cup\}$
σ_K	$\{\emptyset\}$	$\{\emptyset\}$	$\{+,-,\cap\}$	$\{+,-,\cap\}$	$\{+,-,\cap\}$
σ_L	$\{+,-\}$	$\{\emptyset\}$	$\{+,\cup\}$	$\{+,-,\cup\}$	$\{+,-,\cup\}$
				Total	$\{+,-,\cup,\cap\}$

5.3.2 Physical Capital Uncertainty

This subsection analyzes the impacts of physical capital uncertainty σ_K. I don't repeat what we discussed above. As $\partial u/\partial\sigma_K = 0$, the other control variable $c-$

which did not play any role in the previous subsection – will matter to understand the mechanism in this subsection.

We begin with the impacts of σ_K on c, again in the absence of correlation $\eta_{HL} = \eta_{KL} = 0$. From Eq. (5.14), we have

$$\frac{\partial c}{\partial \sigma_K} > 0,$$

as shown in Panel (a) of Fig. 5.5. At first glance, this result may seem at odds; in response to higher uncertainty, consumption is likely to be *reduced* because of a precautionary saving motive. Why does higher physical capital uncertainty induce *more* consumption?[1]

Here, there are *two* conflicting forces that one frequently encounters in macroeconomics: the counteracting influences of income and substitution effects. When agents are less risk-averse ($\phi = \beta < 1$), higher uncertainty boils down to an increase in income and therefore increases (per capita) consumption-capital ratio c/k. At the same time, higher uncertainty lowers the risk associated with savings, thereby inducing less consumption. In the current scenario, the former effect always outbalances the latter, leading to a net *increase* in consumption. See Turnovsky (2000, p.565) for a more in-depth discussion on this mechanism.

Next, we have

$$\frac{\partial \mathcal{G}_K}{\partial \sigma_K} < 0,$$

as shown in panel (c). Higher physical capital uncertainty σ_K lowers the expected growth rate of physical capital $\mathcal{G}_K$. From (5.5), this is intuitive; as consumption increases in response to higher σ_K, savings (and hence physical capital k) are reduced. As a result, the expected growth rate of physical capital decreases. As physical capital uncertainty σ_K and the corresponding standard deviation σ_g^K are positively correlated (panel (b)), this in turn implies the *negative* relationship between growth and uncertainty $\partial \mathcal{G}_K / \partial \sigma_g^K < 0$ (panel (g)).

[1]See Rebelo and Xie (1999, Proposition 7) for more on this point. They solve a stochastic monetary growth model with AK production technology (hence no human capital h) with a parameter restriction of Xie (1991): $\phi = \beta$. They derive the less general, but basically similar expression for consumption in the presence of σ_K (they, however, don't discuss the impacts of σ_K on consumption).

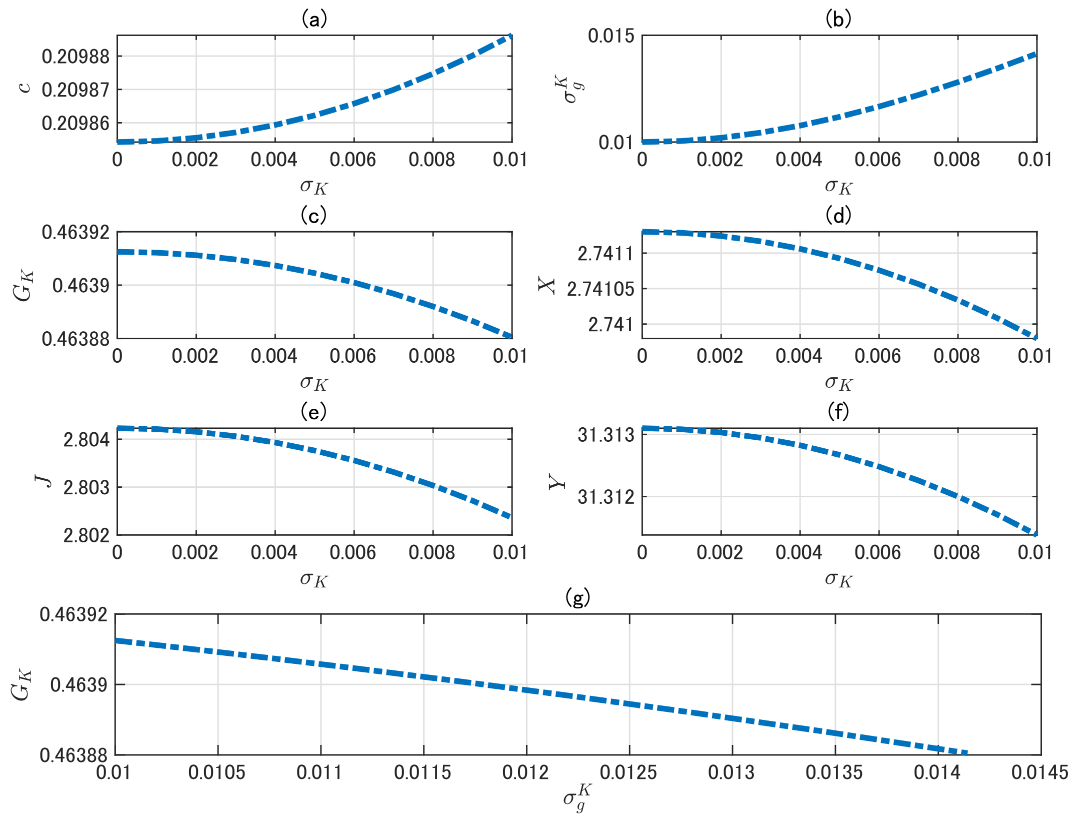

Figure 5.5 The impacts of physical capital uncertainty σ_K when $\eta_{KL} = 0$ and $\eta_{HL} = 0$.

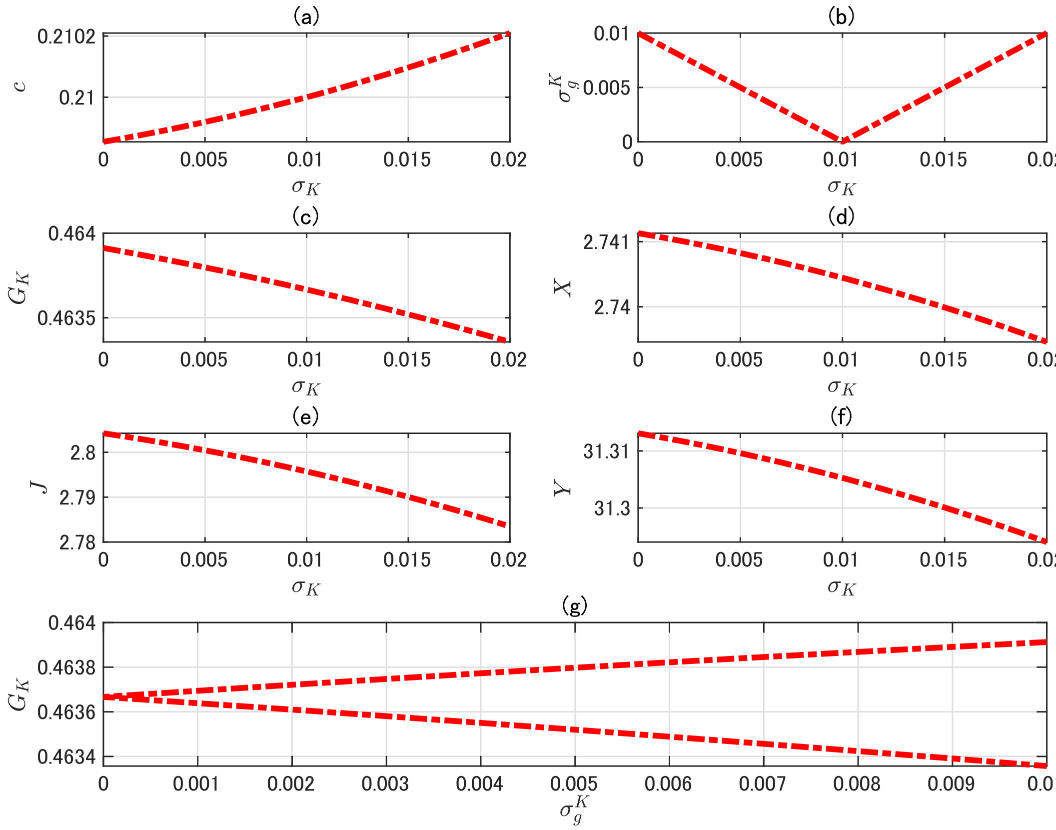

Figure 5.6 The impacts of physical capital uncertainty σ_K when $\eta_{KL} > 0$ and $\eta_{HL} = 0$.

For welfare, we have

$$\frac{\partial J(k,h)}{\partial \sigma_K} < 0,$$

as displayed in panel (e). Here, three (but different from those above) forces are at work; first, as $\partial \mathcal{G}_K/\partial \sigma_K < 0$, higher physical capital uncertainty causes the expected growth rate of physical capital to decrease. Because welfare $J(k,h)$ is increasing in k, this physical capital contraction is detrimental to welfare. Second, we have $\partial \mathbb{X}/\partial \sigma_K < 0$, as shown in panel (d). From a value function (5.10), we know that a decrease in $\mathbb{X}$ means a smaller contribution of k to welfare $J(k,h)$. Third, we have $\partial \mathbb{Y}/\partial \sigma_K < 0$ too, as shown in panel (f). A decrease in $\mathbb{Y}$ means a smaller contribution of h to welfare $J(k,h)$. In total, these three detrimental forces yield a welfare *loss*.

Next, what if $\eta_{KL} > 0$ and $\eta_{HL} = 0$? As Fig. 5.6 displays, the qualitative results again don't change – except for panels (b) and (g). Because of a positive correlation, there is a kink in panel (b).[1] Thus, we find

$$\left.\frac{\partial \mathcal{G}_K}{\partial \sigma_g^K}\right|_{\mathcal{G}_K \gtreqless \hat{\mathcal{G}}_K} \gtreqless 0,$$

implying that, when stochastic processes for the accumulation of physical capital and population dynamics are positively correlated, there exists both positive and negative relationships between growth and uncertainty.

Finally, what if $\eta_{KL} < 0$ and $\eta_{HL} = 0$? As Fig. 5.7 displays, the model generates *nonlinear* dynamics in this case. We begin with our key control variable c,

$$\frac{\partial c}{\partial \sigma_K} \gtreqless 0 \Leftrightarrow \sigma_K \gtreqless -\frac{(1-\beta)\eta_{KL}}{\beta}\sigma_L,$$

shown in panel (a). Why does higher physical capital uncertainty initially reduce consumption, as long as it is moderate? To see this, note that

[1] As shown in Appendix 5.B, we have

$$\frac{\partial \sigma_g^K}{\partial \sigma_K} \gtreqless 0 \Leftrightarrow \sigma_K \gtreqless \eta_{KL}\sigma_L.$$

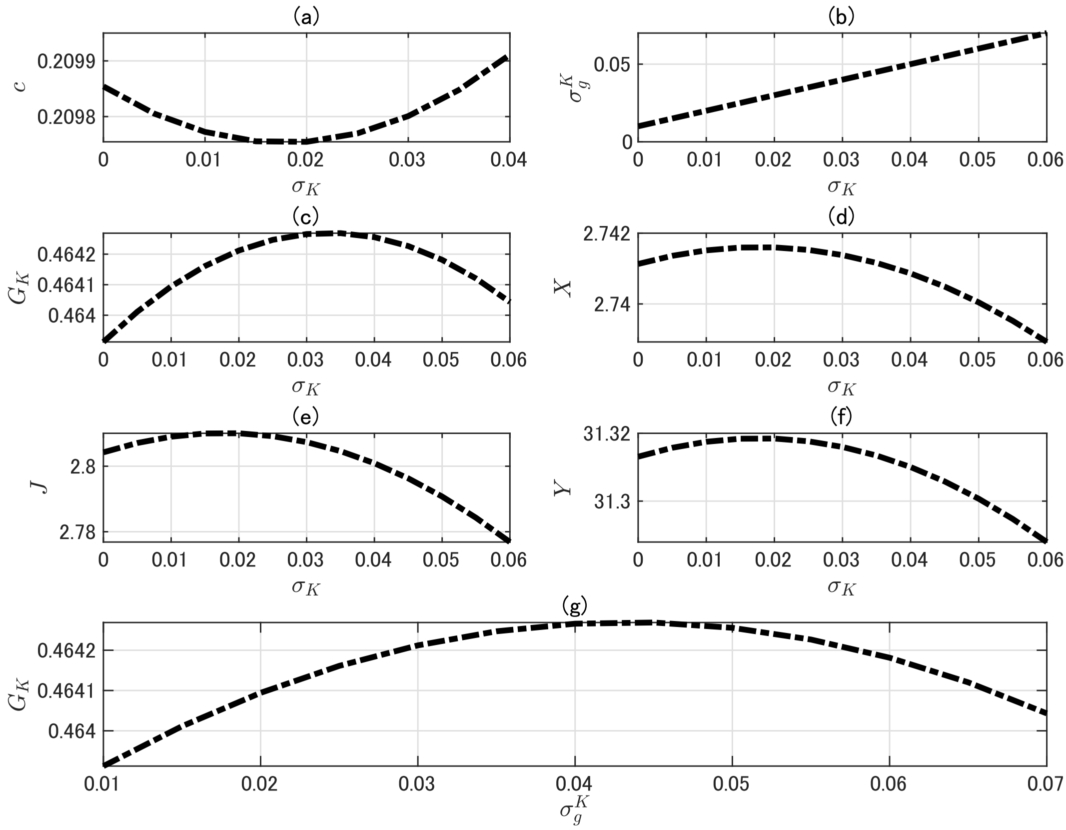

Figure 5.7 The impacts of physical capital uncertainty σ_K when $\eta_{KL} < 0$ and $\eta_{HL} = 0$.

$$\frac{\partial c}{\partial \eta_{KL}} > 0,$$

that is, a higher correlation between physical capital accumulation and population dynamics induces more consumption. With this in mind, we should think of what $\eta_{KL} < 0$ means. This case may represent *capital dilution*: the negative effect of population growth on per capita physical capital. Put differently, higher population growth dilutes the per capita physical capital stock k more quickly than usual ($\eta_{KL} = 0$); so people have to save *more* (and consume *less*) to prevent physical capital stock from getting lower. Due to this force, in contrast to panel (a) in Fig. 5.6, there is an initial *decrease* in consumption. This impact being reflected, we see a *U-shaped* pattern of per capita consumption c in panel (a) of Fig. 5.7.

Understanding this, we can interpret the rest of dynamics. As we have

$$\frac{\partial \mathcal{G}_K}{\partial \sigma_K} \gtrless 0 \Leftrightarrow \sigma_K \lesseqgtr -\frac{(1-\beta(1-\beta))\eta_{KL}}{\beta(1-\beta)}\sigma_L,$$

there is an inverted U-shaped relationship between the expected growth rate of physical capital and its uncertainty σ_K, as displayed in panel (c). Remember that, in case of nonnegative η_{KL}, c and $\mathcal{G}_K$ go to the opposite direction. Armed with this insight, it is natural that we see an inverted U-shaped pattern of $\mathcal{G}_K$ when c draws a U-shaped dynamic path. Therefore, with panel (b), we can immediately see

$$\left.\frac{\partial \mathcal{G}_K}{\partial \sigma_g^K}\right|_{\sigma_g^K \lesseqgtr \sigma_g^{K*}} \gtrless 0,$$

as in panel (g). There is an *inverted U-shaped* association between growth and uncertainty. For welfare $J(k,h)$ (panel (e)), we can see that its hump-shaped pattern is because of nonlinear response of $\mathbb{X}$ (panel (d)) and $\mathbb{Y}$ (panel (f)) to increases in σ_K.

The findings of this subsection can be summarized as follows:

Proposition 5.2. *When there is a positive or no correlation between popula-*

tion dynamics and physical capital accumulation, the model predicts a positive or negative relationship between growth and uncertainty. In contrast, when they are negatively correlated, the model predicts an inverted U-shaped relationship between growth and uncertainty.

Put differently, physical capital uncertainty can replicate *three* of four patterns suggested by empirical studies. It is summarized in the second row of columns (i) to (iv) of Table 5.1.

5.3.3 Demographic Uncertainty

This final subsection analyzes the impacts of demographic uncertainty σ_L. Since we have $\partial c/\partial \sigma_L \neq 0$ and $\partial u/\partial \sigma_L \neq 0$, at the outset, we predict that our analysis will necessarily be involved. Having understood the main mechanisms above and the meaning of η_{HL} and η_{KL}, however, we are equipped with basic insights to grasp the main points.

First, we have

$$\frac{\partial c}{\partial \sigma_L} < 0.$$

As displayed in panel (a) of Fig. 5.8, higher demographic uncertainty reduces consumption. As Bucci et al. (2011) explain in the context of technology shocks, here, a precautionary saving motive is at work. A rise in uncertainty leads people to increase their precautionary saving; so it reduces their consumption expenditure (Bloom, 2014). This implies that $\mathcal{G}_K$ increases in response to higher σ_L, as shown in panel (f). Because σ_L and σ_g^K are positively correlated (panel (b)), we must have

$$\frac{\partial \mathcal{G}_K}{\partial \sigma_g^K} > 0,$$

meaning a *positive* relationship between growth and uncertainty.[1]

Next, we have

[1]The diagrams below have basically 9 panels. I won't show an obvious one so that they won't be busy.

$$\frac{\partial u}{\partial \sigma_L} < 0,$$

as displayed in panel (d). In response to higher σ_L, people tend to spend more of their time in learning. This allocation of time increases human capital stock, hence generating a positive relationship between $\mathcal{G}_H$ and σ_L (panel (e)). This, in turn, implies (via panel (c)),

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_g^H} > 0,$$

meaning again a *positive* link between growth and uncertainty. The accumulation of human capital, at the same time, improves welfare $J(k, h)$ (panel (g)) together with $\partial \mathbb{X}/\partial \sigma_L > 0$ (panel (h)) and $\partial \mathbb{Y}/\partial \sigma_L > 0$ (panel (i)).

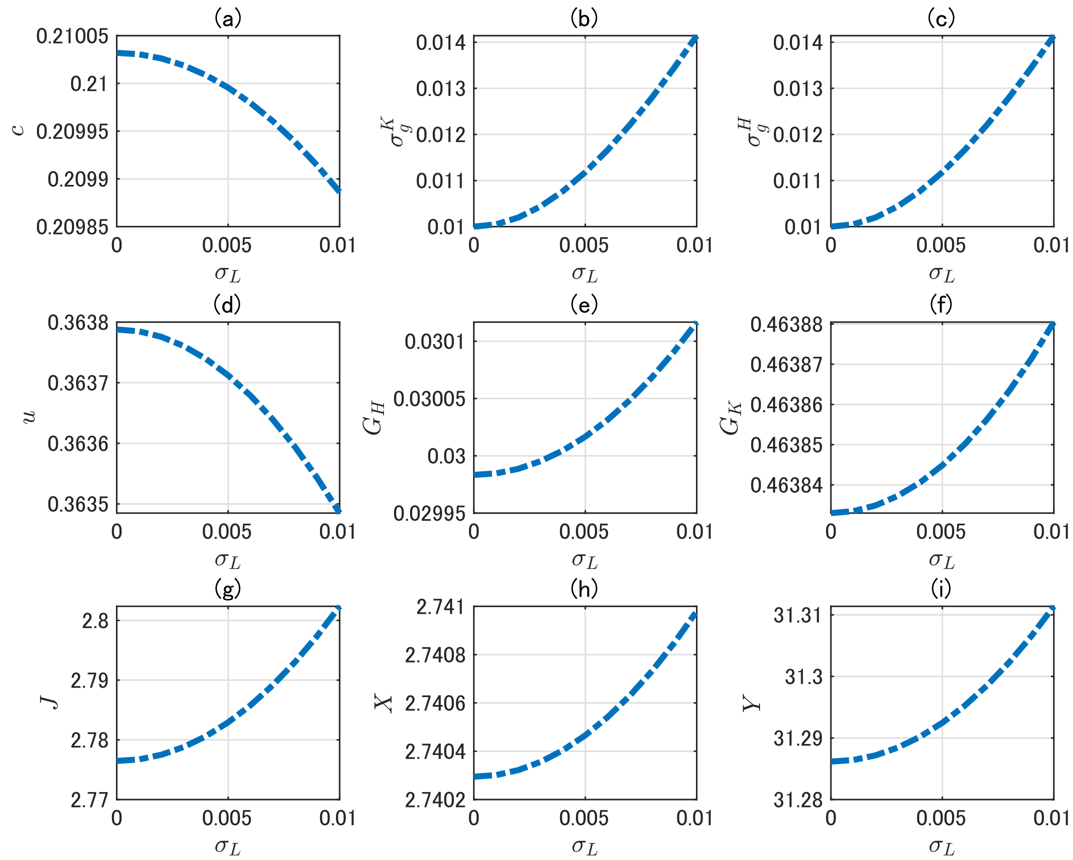

Figure 5.8 The impacts of demographic shocks σ_L when $\eta_{HL} = 0$ and $\eta_{KL} = 0$.

This completes the discussion of σ_L in the *absence* of correlation. In what follows, for brevity, I shall divide this subsection into two parts: the first only with η_{HL} (and $\eta_{KL} = 0$) and the second only with η_{KL} (and $\eta_{HL} = 0$). Analysis of the case with both $\eta_{HL} \neq 0$ *and* $\eta_{KL} \neq 0$ is so complex that I put it in Appendix 5.C.

5.3.3.1 Correlation with Human Capital Uncertainty Only

The case of $\eta_{HL} > 0$ is displayed in Fig. 5.9. I focus on the nonlinear relation, such as

$$\frac{\partial u}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \lessgtr \frac{\alpha \eta_{HL}}{1+\alpha}\sigma_H,$$

shown in panel (d). We know that an inverted U-shaped dynamics for u means a U-shaped dynamics for $\mathcal{G}_H$,

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \gtrless \frac{1-\alpha(1-\alpha)\eta_{HL}}{2-\alpha(1-\alpha)}\sigma_H,$$

as displayed in panel (e). Moreover, we also know that

$$\frac{\partial J(k,h)}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \gtrless \frac{\alpha \eta_{HL}}{1+\alpha}\sigma_H,$$

as shown in panel (g). For growth and uncertainty, panels (b), (c), (e), and (f) indicate that

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_g^H} \gtrless 0, \quad \frac{\partial \mathcal{G}_K}{\partial \sigma_g^K} > 0.$$

Therefore, when $\eta_{HL} > 0$, there exists a positive or negative link between growth and uncertainty.[1] The opposite case of $\eta_{HL} < 0$ is displayed in Fig. 5.10. In this case, as clear from panels (b), (c), (e), (f), and (j), we have

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_g^H} > 0, \quad \frac{\partial \mathcal{G}_K}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \gtrless \frac{\eta_{HL}\alpha^2\beta u^{\alpha-1}}{(1-\alpha)(b(2-\beta(1-\beta))-\alpha\beta u^{\alpha-1})}\sigma_H,$$

that is, we see either a positive or U-shaped relation.

[1] To make Fig. 5.9 visible, the relation between $\mathcal{G}_H$ and σ_g^H is not shown.

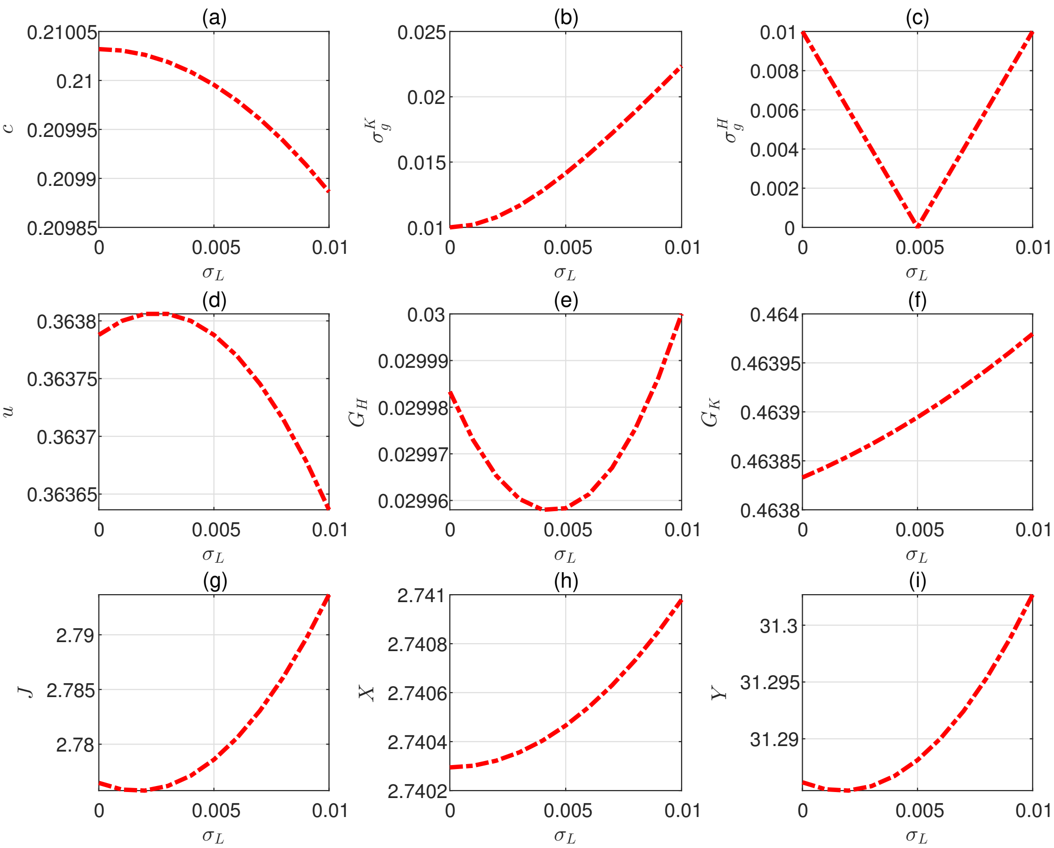

Figure 5.9 The impacts of demographic uncertainty σ_L when $\eta_{HL} > 0$ and $\eta_{KL} = 0$.

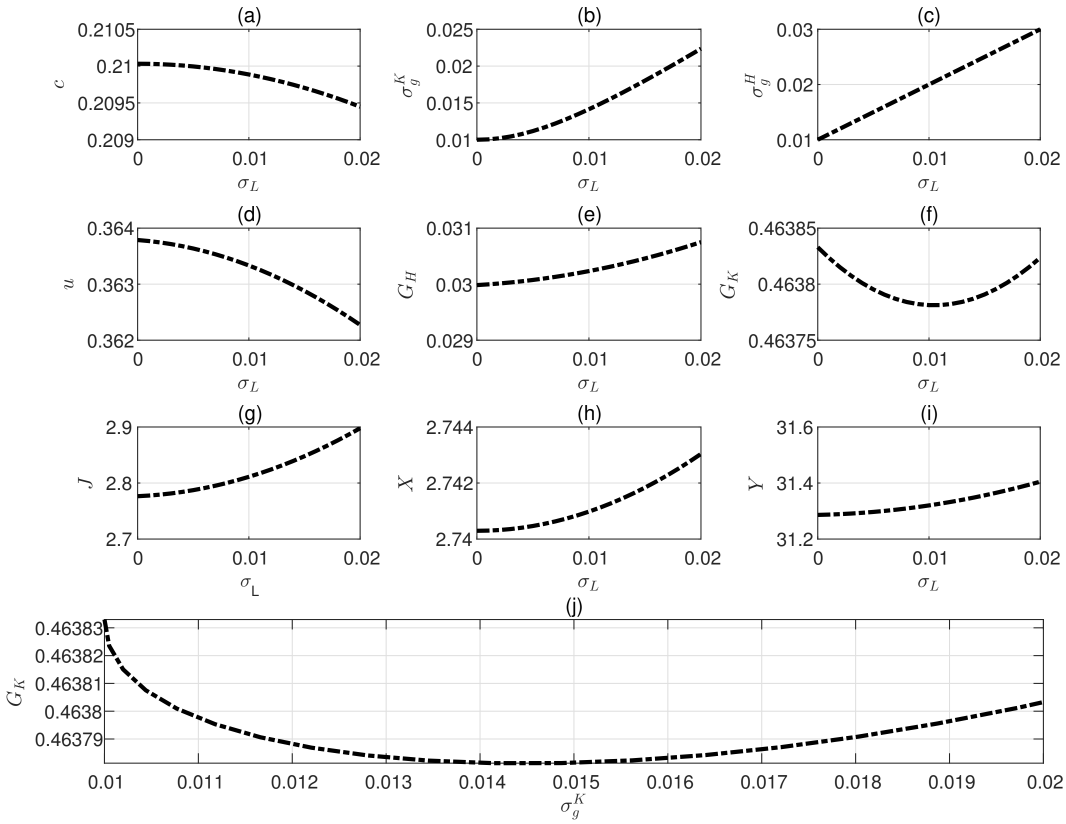

Figure 5.10 The impacts of demographic uncertainty σ_L when $\eta_{HL} < 0$ and $\eta_{KL} = 0$.

5.3.3.2 Correlation with Physical Capital Uncertainty Only

The case of $\eta_{KL} > 0$ is displayed in Fig. 5.11. In this case too, we see a U-shaped pattern for $\mathcal{G}_K$ (panel (f)):

$$\frac{\partial \mathcal{G}_K}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \gtrless \frac{\eta_{KL}\sigma_K b(1-\alpha)(1-\beta(1-\beta))}{(1-\alpha)(b(2-\beta(1-\beta))-\alpha\beta u^{\alpha-1})}.$$

The other variable that exhibits nonlinear dynamics is c:

$$\frac{\partial c}{\partial \sigma_L} \gtrless 0 \Leftrightarrow \sigma_L \gtrless \frac{(1-\beta)\eta_{KL}}{2-\beta}\sigma_K,$$

as shown in panel (a) (not visible). For our purpose, panels (b), (c), (e), (f), and (j) make it clear that

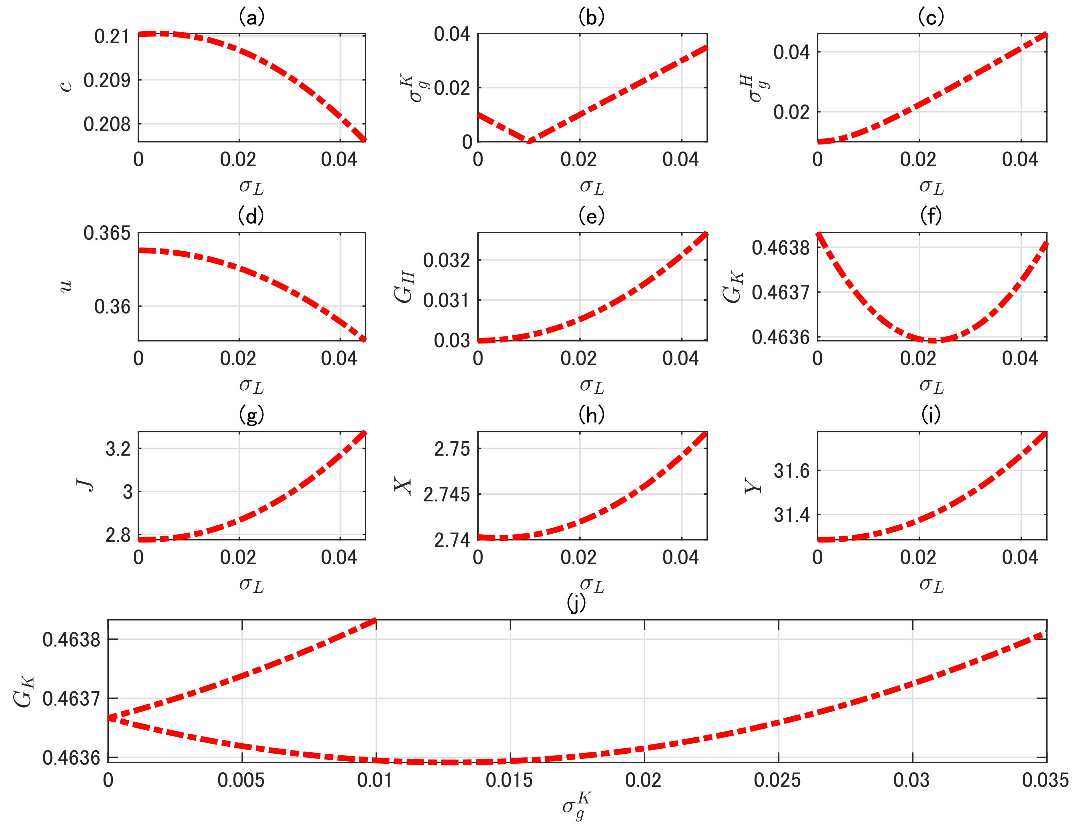

Figure 5.11 The impacts of demographic uncertainty σ_L when $\eta_{KL} > 0$ and $\eta_{HL} = 0$.

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_g^H} > 0,$$

so there is a *positive* relationship between growth and uncertainty.

Finally, the case of $\eta_{KL} < 0$ is displayed in Fig. 5.12. This case, however, only generates a linear relationship. Thus, we have

$$\frac{\partial \mathcal{G}_H}{\partial \sigma_g^H} > 0, \quad \frac{\partial \mathcal{G}_K}{\partial \sigma_g^K} > 0.$$

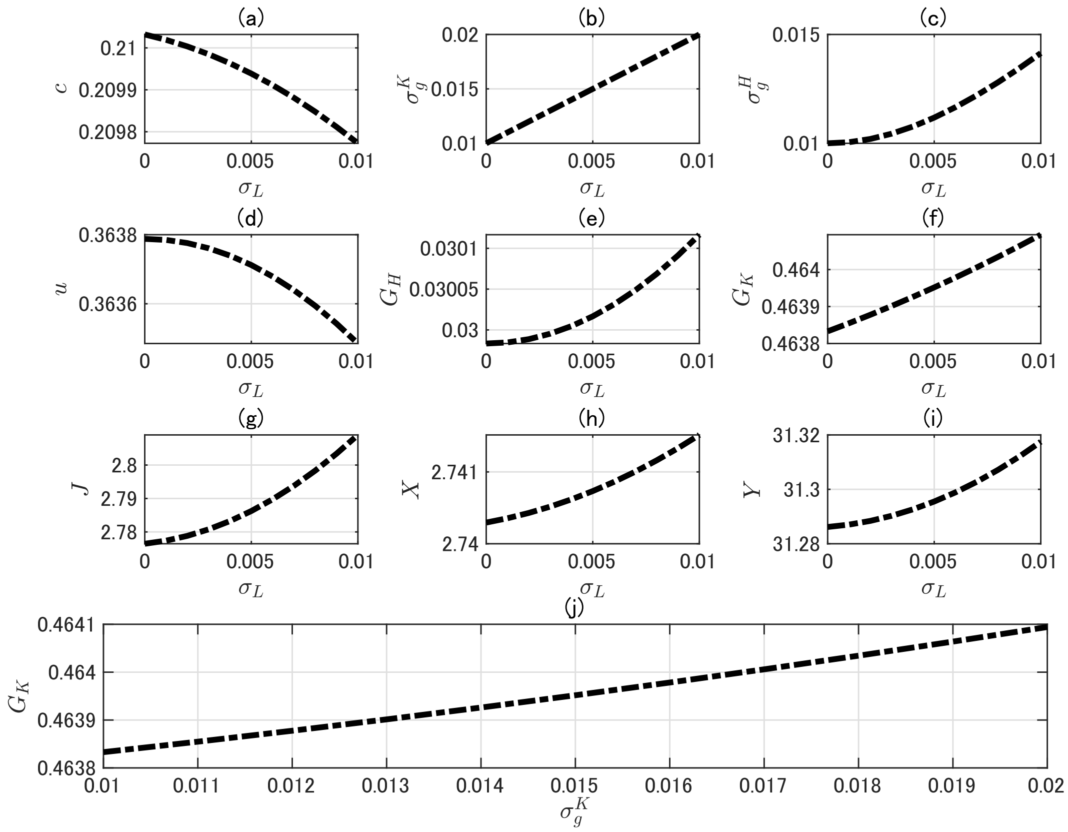

Figure 5.12 The impacts of demographic uncertainty σ_L when $\eta_{KL} < 0$ and $\eta_{HL} = 0$.

The findings of this subsection can be summarized as follows:

Proposition 5.3. *In general, there is a positive relationship between growth and uncertainty. When the population dynamics and accumulation of human capital are positively correlated, however, there exists an inverted U-shaped relationship.*

Demographic uncertainty thus can replicate *three* of four patterns suggested by empirical studies. It is summarized in the third row of columns (i) to (iv) in Table 5.1.

5.4 Aggregate Uncertainty and Output Growth

So far, we have analyzed the link between three types of uncertainty and expected growth rates of *capital*. Given the nature of empirical studies, this final section briefly examines the relationship between *aggregate* uncertainty and *output* growth. As in Appendix 5.C, even with the closed-form solution, their analysis is not straightforward. Thus, after deriving analytical formulas, I provide numerical examples. Since we already know the channels through which uncertainty affects growth from the previous section, I keep the exposition minimum.

First, applying Itô's lemma to the production function $y(t) = (uh(t))^\alpha k(t)^\beta$, I obtain[1]

$$\begin{aligned} dy(t) &= \frac{\partial y(t)}{\partial h(t)} dh(t) + \frac{\partial y(t)}{\partial k(t)} dk(t) + \frac{\partial^2 y(t)}{\partial h(t) \partial k(t)} (dh(t))(dk(t)) + \frac{1}{2} \frac{\partial^2 y(t)}{\partial h(t)^2} (dh(t))^2 \\ &+ \frac{1}{2} \frac{\partial^2 y(t)}{\partial k(t)^2} (dk(t))^2 \\ &= \alpha \frac{y(t)}{h(t)} dh(t) + \beta \frac{y(t)}{k(t)} dk(t) + \alpha\beta \frac{y(t)}{h(t)k(t)} (dh(t))(dk(t)) - \frac{1}{2}\alpha(1-\alpha) \frac{y(t)}{h(t)^2} (dh(t))^2 \\ &- \frac{1}{2}\beta(1-\beta) \frac{y(t)}{k(t)^2} (dk(t))^2. \end{aligned}$$

Thus, I get

$$\begin{aligned} \frac{dy(t)}{y(t)} &= \alpha \frac{dh(t)}{h(t)} + \beta \frac{dk(t)}{k(t)} + \alpha\beta \left(\frac{dh(t)}{h(t)} \right) \left(\frac{dk(t)}{k(t)} \right) - \frac{1}{2}\alpha(1-\alpha) \left(\frac{dh(t)}{h(t)} \right)^2 \\ &- \frac{1}{2}\beta(1-\beta) \left(\frac{dk(t)}{k(t)} \right)^2. \end{aligned}$$

Next, taking expectations of both sides and dividing them by dt, after some algebra, I have the expected growth rate of output $\mathcal{G}_Y$:

$$\mathcal{G}_Y \equiv \mathbb{E}\left(\dot{y}(t)/y(t)\right) = \alpha \mathcal{G}_H + \beta \mathcal{G}_K + \alpha\beta \mathcal{G}_H \mathcal{G}_K - \frac{\alpha(1-\alpha)}{2} \mathcal{G}_H^2 - \frac{\beta(1-\beta)}{2} \mathcal{G}_K^2,$$

where $\dot{y}(t) \equiv dy(t)/dt$. Due to the product term $\mathcal{G}_H \mathcal{G}_K$ and quadratic terms $\mathcal{G}_i^2$,

[1] Note that u is constant after optimization.

comparative statics by hand is uninspiring.

Finally, let σ_g^Y denote the standard deviation of $\mathcal{G}_Y$. Using the same technique in Appendix 5.B, I calculate it as follows:

$$
\begin{aligned}
(\sigma_g^Y)^2 &\equiv \left(\frac{dy(t)}{y(t)} - \mathbb{E}\left(\frac{dy(t)}{y(t)}\right)\right)^2 = \mathbb{E}[\alpha(\sigma_H dz_H - \sigma_L dz_L) + \beta(\sigma_K dz_K - \sigma_L dz_L)]^2 \\
&= \alpha^2\left(\sigma_H^2 - 2\eta_{HL}\sigma_H\sigma_L + \sigma_L^2\right) + 2\alpha\beta\left(\sigma_L^2 - \eta_{HL}\sigma_H\sigma_L - \eta_{KL}\sigma_K\sigma_L\right) \\
&+ \beta^2\left(\sigma_K^2 - 2\eta_{KL}\sigma_K\sigma_L + \sigma_L^2\right).
\end{aligned}
$$

Therefore, I find

$$
\begin{aligned}
\sigma_g^Y &= \sqrt{\alpha^2\left(\sigma_g^H\right)^2 + 2\alpha\beta\left(\sigma_L^2 - \eta_{HL}\sigma_H\sigma_L - \eta_{KL}\sigma_K\sigma_L\right) + \beta^2\left(\sigma_g^K\right)^2} \\
&= \sqrt{(\alpha+\beta)\left(\alpha\left(\sigma_g^H\right)^2 + \beta\left(\sigma_g^K\right)^2\right) - \alpha\beta\left(\sigma_H^2 + \sigma_K^2\right)}.
\end{aligned}
$$

Using these key formulas, we can now see how σ_i-induced changes in σ_g^Y (that I illustrated in Fig. 5.1) affect $\mathcal{G}_Y$: the growth-uncertainty nexus that has substantially been analyzed in empirical studies. The results are shown in self-explanatory Figs. 5.13 to 5.21 and, for convenience, summarized in the final column (v) of Table 5.1. For each uncertainty σ_i, I have 18 ($= 6 \times 3$) figures. I just note that we have all four patterns, hence successfully accounting for why the results of empirical studies are surprisingly mixed.

5.5 Concluding Remarks

Will higher uncertainty speed up or slow down growth? Empirical studies suggest four links between growth and uncertainty – their relationship is (i) negative (ii) positive (iii) U-shaped and (iv) inverted U-shaped. To account for these conflicting findings, I analytically analyze a two-sector stochastic endogenous growth model of Uzawa (1965) and Lucas (1988). As illustrated in Fig. 5.1, my model features three types of uncertainty: demographic uncertainty, human capital uncertainty, and physical capital uncertainty. Assuming the correlation between the first and second, and the first and third, I examine the impacts of uncertainty on

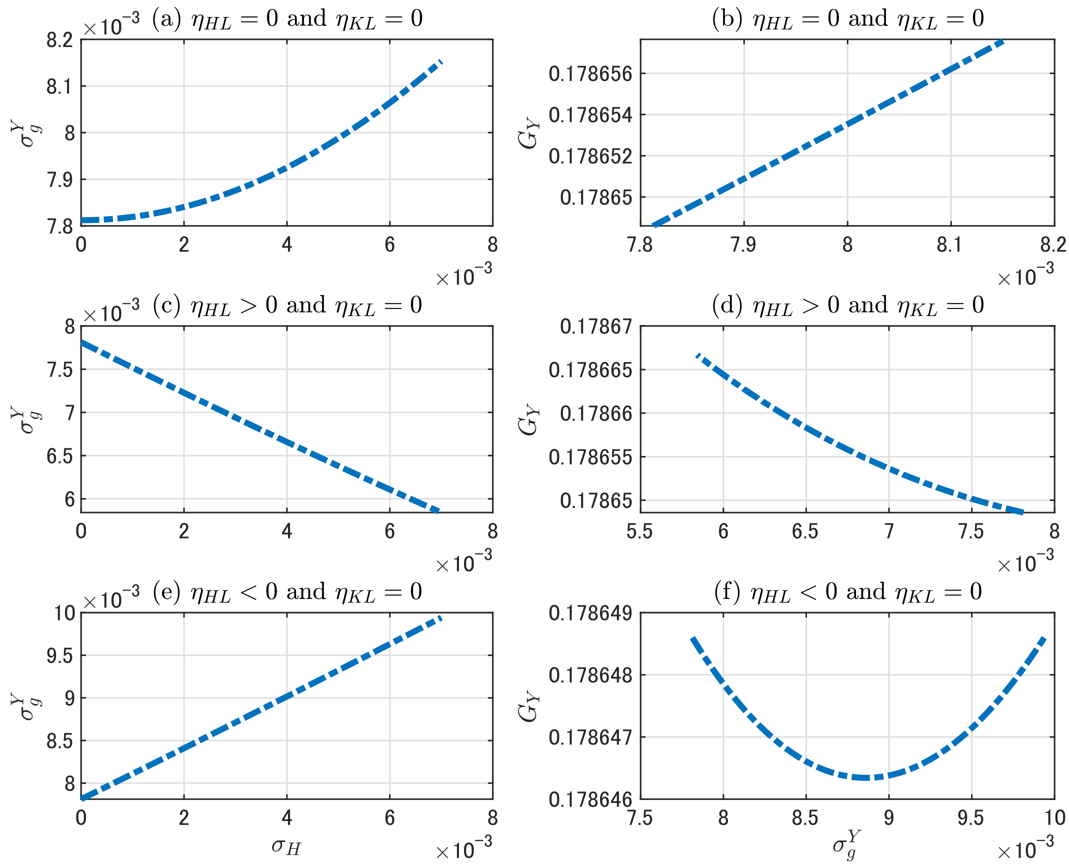

Figure 5.13 Aggregate uncertainty and output growth with σ_H.

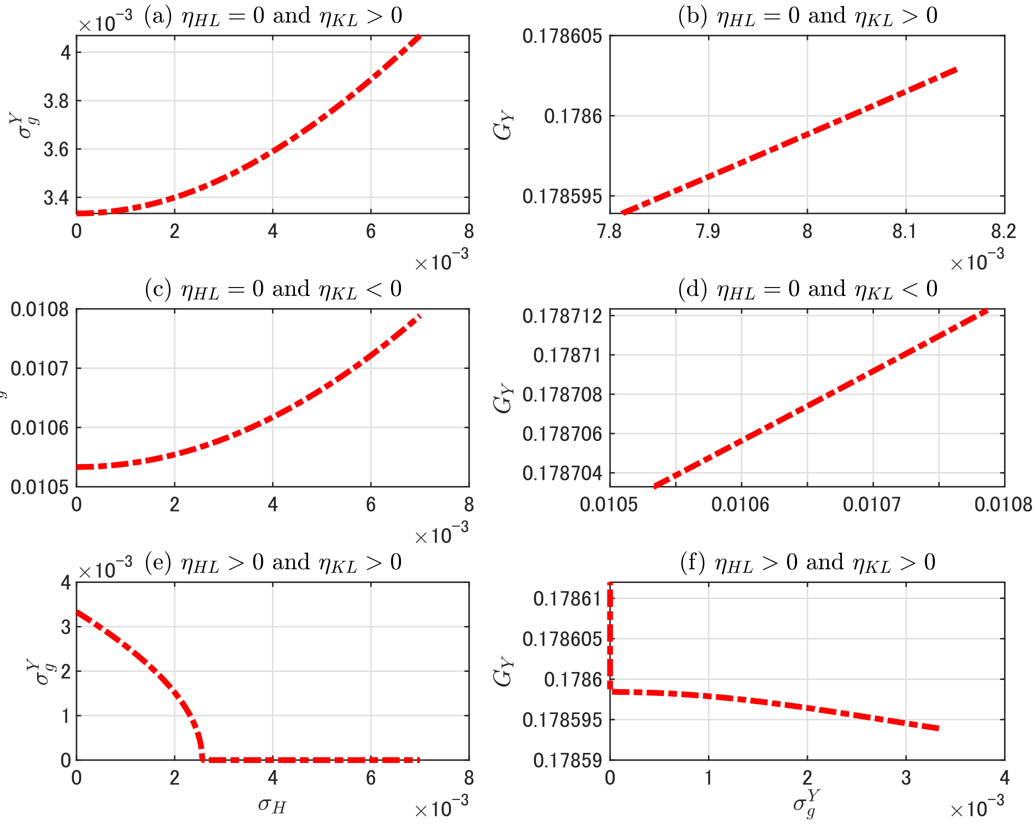

Figure 5.14 Aggregate uncertainty and output growth with σ_H.

growth. Overall, my model can predict all four patterns (and in particular, *non-linear* patterns); hence shedding analytical light on divergent empirical evidence on the growth-uncertainty nexus.

Contrary to some related studies such as Smith (1996), Femminis (2001), Bucci et al. (2011), Posch (2011), Posch and Wälde (2011), and Hiraguchi (2013), I focus on *types* of uncertainty, instead of directly looking at *aggregate* uncertainty. As I show, each uncertainty has different impacts on major macroeconomic variables, especially on growth. Existing empirical studies, however, entirely focus on aggregate uncertainty. According to my findings, it may be useful for empirical studies to examine *input-level* uncertainty and their correlation; for example, rates of population growth and their uncertainty, human capital growth and its uncertainty, and physical capital growth and its uncertainty, and how they are correlated.

Of course, the measurement of such variables is extremely difficult. How do we measure human capital uncertainty? Despite this kind of difficulty, my findings speak to the need for such empirical investigation. When it is accomplished,

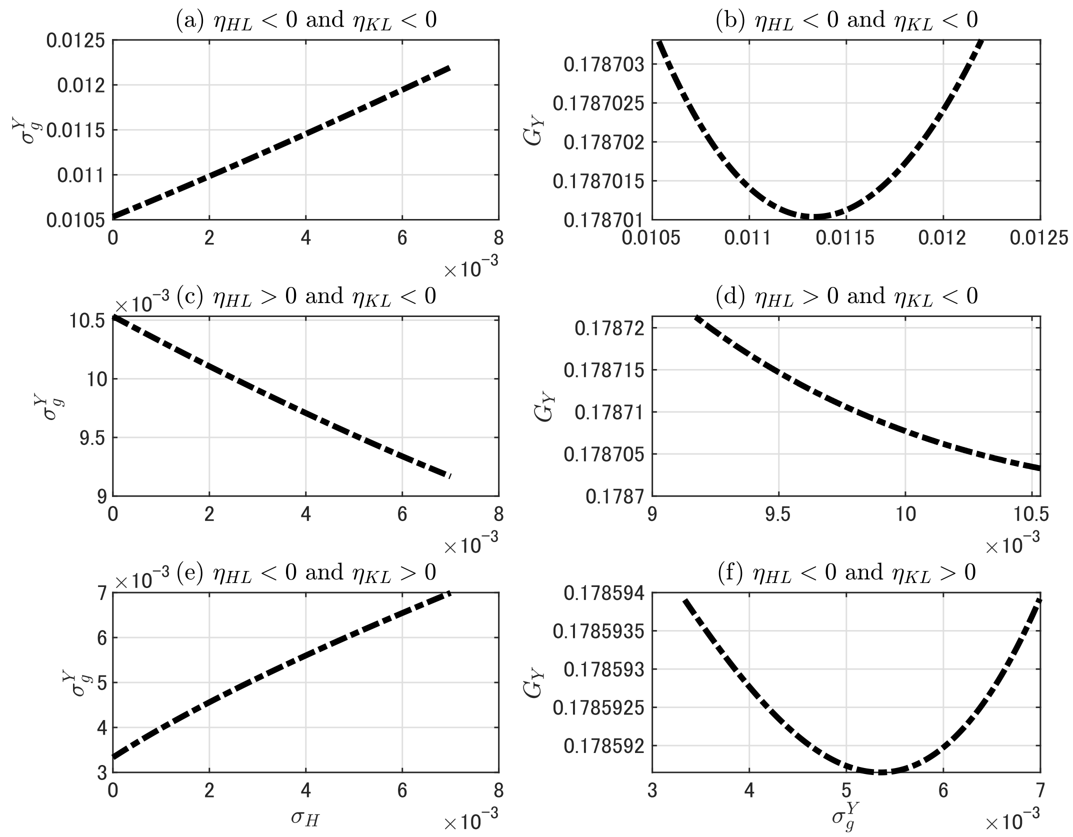

Figure 5.15 Aggregate uncertainty and output growth with σ_H.

we may no longer have empirical ambiguities in the growth-uncertainty nexus discussed in Ch. 1. According to Ramey and Ramey (1995), a short-run stabilization policy *accelerates* the long-run growth. Is it really true? When we understand the true relationship between growth and uncertainty, policymakers can stabilize or destabilize the short-run business cycles with confidence, to maximize economic growth – and possibly to maximize the welfare of people. My findings would be of help in the design of such policy.

This chapter has two limitations. First, I need the parameter restriction of Xie (1991) to analytically solve the model. Though closed-form solutions help us clearly understand an intuition and underlying mechanism, I cannot see how results alter when people are very risk-averse. As such, I need to supplement the formal analytics with numerical simulations of HJB equations. Second, this chapter only undertakes one-way analysis as in the empirical literature; it analyzes how *uncertainty* affects growth, but it doesn't analyze how *growth* affects uncertainty. This two-way analysis may provide insights into better policymaking

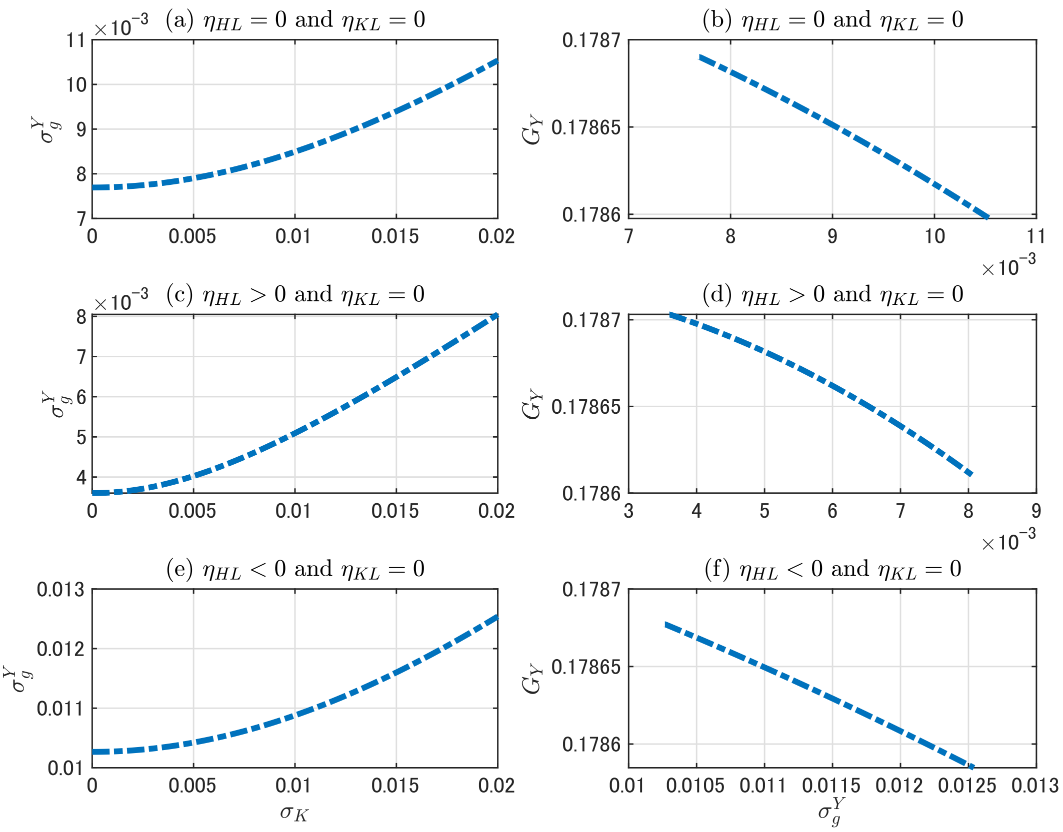

Figure 5.16 Aggregate uncertainty and output growth with σ_K.

under uncertainty, but it is beyond the scope of this chapter.

5.A Value Function

This Appendix shows how to find the functional form of the value function in Sect. 2. The presentation is based on Appendix A of Bucci et al. (2011). I postulate a tentative form:

$$J(k, h) = T_X k^{\theta_1} + T_Y h^{\theta_2} + T_Z,$$

where T_X, T_Y, T_Z, θ_1, and θ_2 are unknown constants to be determined. Relevant partials are $J_K = \theta_1 T_X k^{\theta_1 - 1}$, $J_{KK} = \theta_1(\theta_1 - 1)T_X k^{\theta_1 - 2}$, $J_H = \theta_2 T_Y h^{\theta_2 - 1}$, and $J_{HH} = \theta_2(\theta_2 - 1)T_Y h^{\theta_2 - 2}$. First, substitute these into the maximized HJB equation (5.9) in the main text. Second, choose $\theta_1 = 1 - \beta$, $\theta_2 = \alpha$, and impose $\phi = \beta$. We can then find the closed-form representation of a value function (5.10), together with constants (5.11), (5.12), and (5.13). The proof of optimality con-

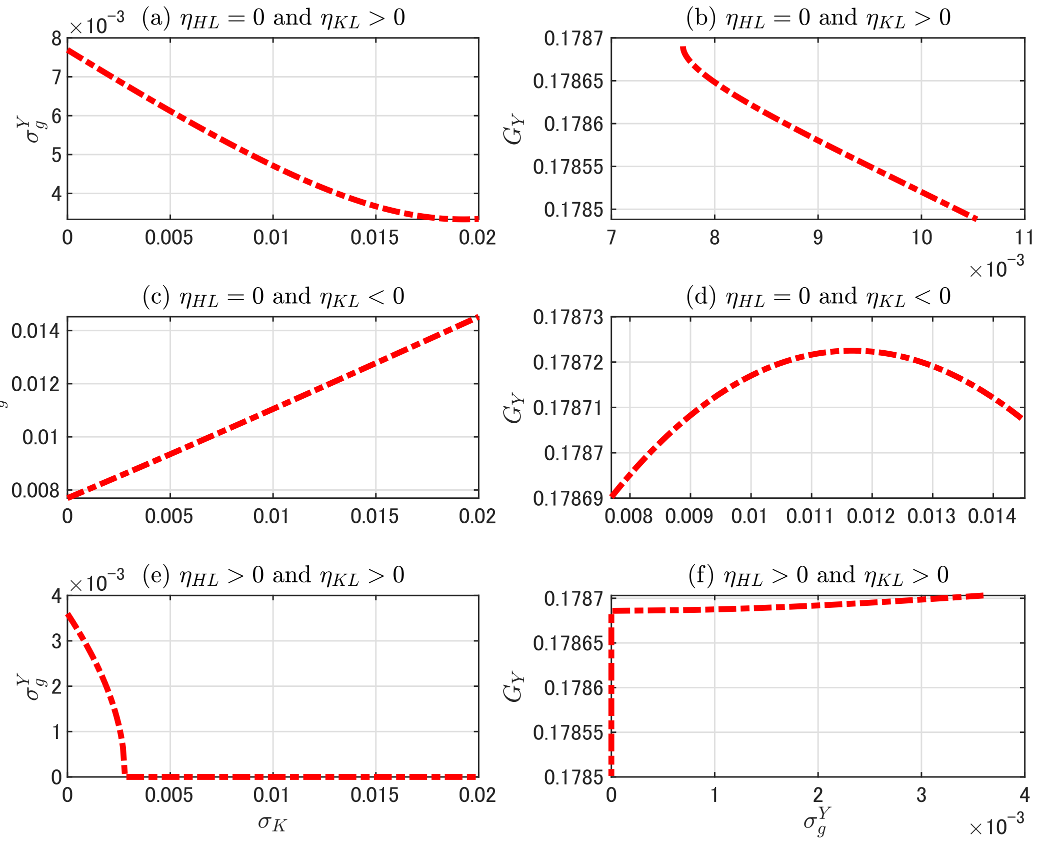

Figure 5.17 Aggregate uncertainty and output growth with σ_K.

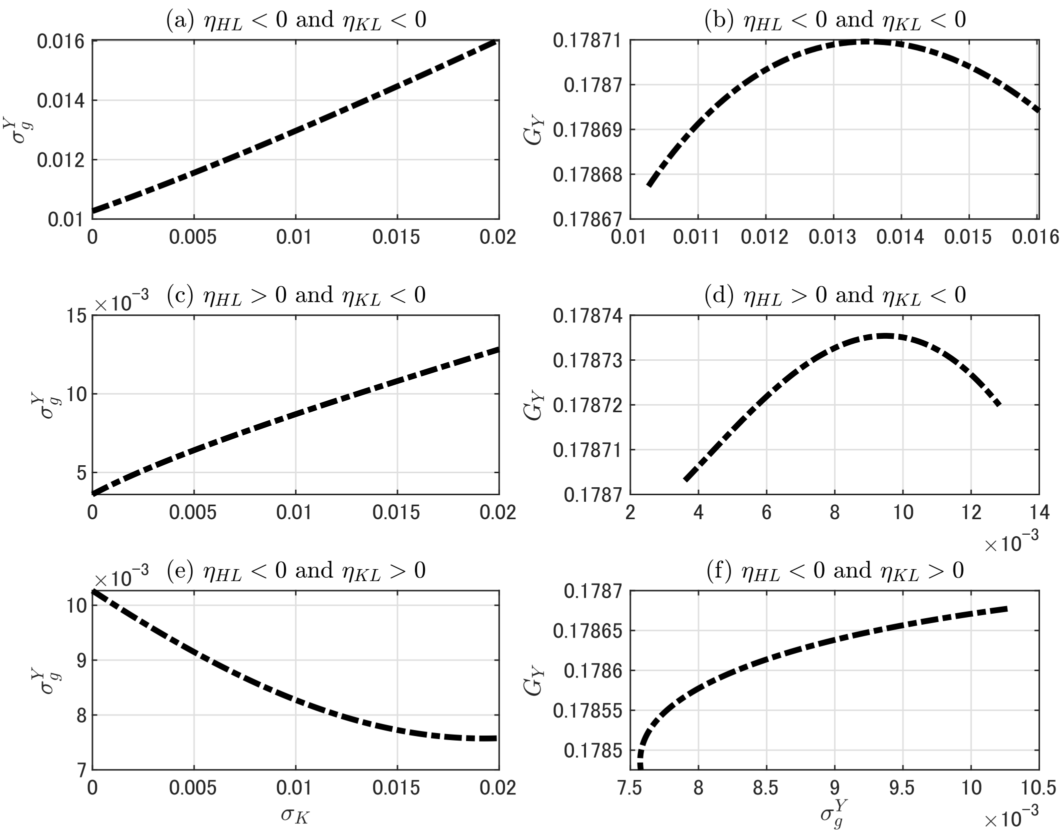

Figure 5.18 Aggregate uncertainty and output growth with σ_K.

ditions requires a verification theorem; see Appendix A of Bucci et al. (2011) or Chang (2004, Ch.4) for details. The TVC is

$$\lim_{t\to\infty} \mathbb{E}[e^{-\rho t}k(t)^{1-\beta}] = \lim_{t\to\infty} \mathbb{E}[e^{-\rho t}h(t)^{\alpha}] = 0.$$

Its proof, however, is so involved that I refer a reader to Appendix B of Hiraguchi (2013) for an excellent demonstration. In essence, this TVC is satisfied when $u \in (0, 1)$.

5.B Standard Deviation

Does uncertainty σ_i (for $i = H, K, L$) correspond with the standard deviation σ_g^i shown in figures of Ch. 1? This Appendix shows that the answer is yes, by deriving expressions for the standard deviation of growth. Let σ_g^H, σ_g^K, and σ_g^L denote the standard deviation of $\mathcal{G}_H$, the standard deviation of $\mathcal{G}_K$, and the standard deviation of expected rate of population growth, respectively. I begin

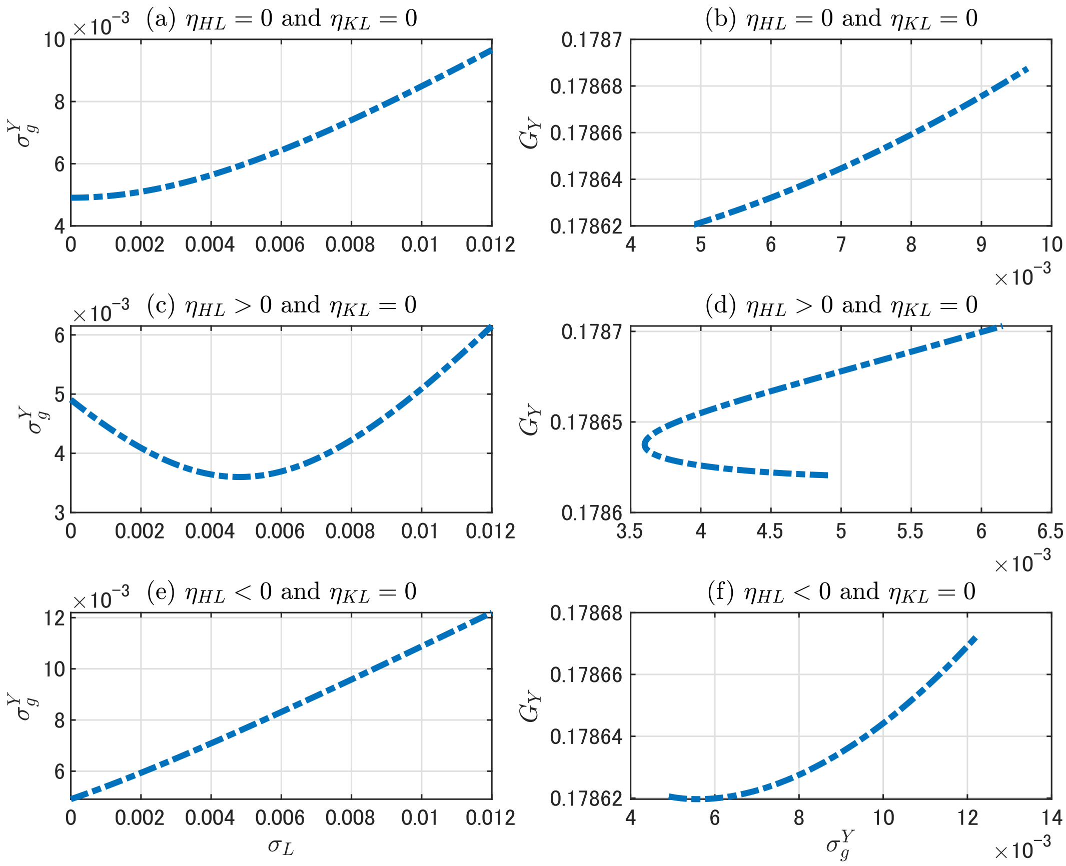

Figure 5.19 Aggregate uncertainty and output growth with σ_L.

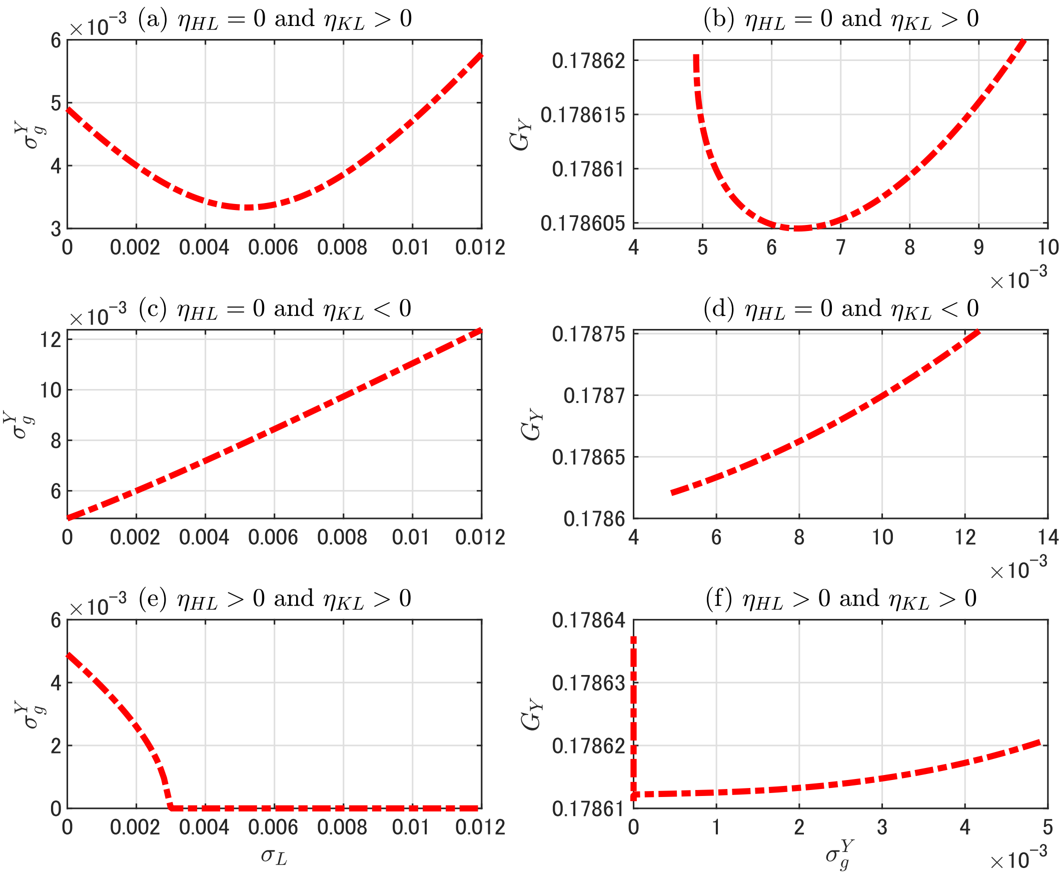

Figure 5.20 Aggregate uncertainty and output growth with σ_L.

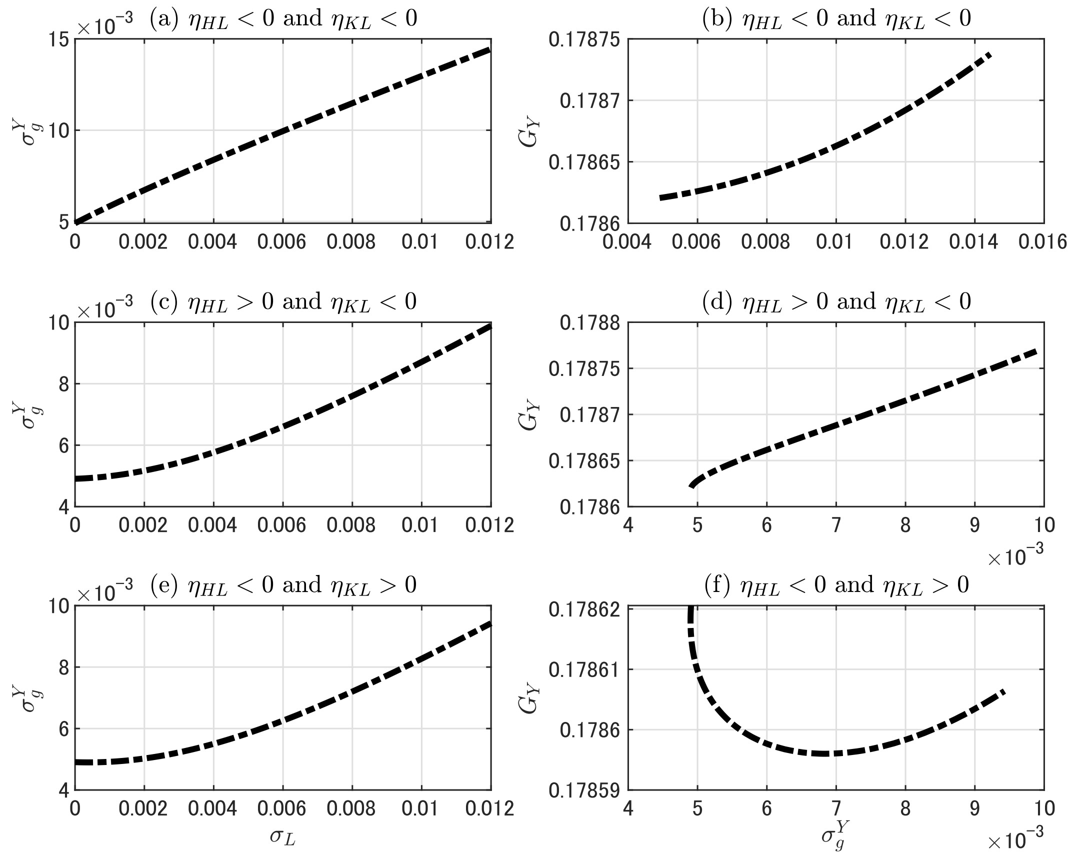

Figure 5.21 Aggregate uncertainty and output growth with σ_L.

with σ_g^H. Noting that the variance of $\mathcal{G}_H$ is $(\sigma_g^H)^2$, we can calculate it as follows:

$$
\begin{aligned}
(\sigma_g^H)^2 &= \mathbb{E}\left(\frac{dh}{h} - \mathbb{E}\left(\frac{dh}{h}\right)\right)^2 = \mathbb{E}(\sigma_H dz_H - \sigma_L dz_L)^2 \\
&= \mathbb{E}\left(\sigma_H^2 (dz_H)^2 - 2\sigma_H \sigma_L (dz_H)(dz_L) + \sigma_L^2 (dz_L)^2\right) \\
&= \sigma_H^2 \underbrace{\mathbb{E}(dz_H^2)}_{=1} - 2\sigma_H\sigma_L \underbrace{\mathbb{E}(dz_H \cdot dz_L)}_{=\eta_{HL}} + \sigma_L^2 \underbrace{\mathbb{E}(dz_L^2)}_{=1} \\
&= \sigma_H^2 - 2\eta_{HL}\sigma_H\sigma_L + \sigma_L^2,
\end{aligned}
$$

where

$$
\eta_{HL} = \underbrace{Cov(dz_H, dz_L)}_{\text{Covariance}} = \mathbb{E}(dz_H \cdot dz_L) - \underbrace{\mathbb{E}(dz_H)}_{=0} \cdot \underbrace{\mathbb{E}(dz_L)}_{=0}.
$$

Therefore, we get

$$
\sigma_g^H = \sqrt{\sigma_H^2 - 2\eta_{HL}\sigma_H\sigma_L + \sigma_L^2}.
$$

By the same token, one can show

$$
\sigma_g^K = \sqrt{\sigma_K^2 - 2\eta_{KL}\sigma_K\sigma_L + \sigma_L^2},
$$

and $\sigma_g^L = \sigma_L$. These expressions demonstrate that, in the absence of correlation ($\eta_{HL} = \eta_{KL} = 0$) or when two stochastic processes are negatively correlated ($\eta_{HL} < 0$ and/or $\eta_{KL} < 0$), the standard deviation σ_g^i and uncertainty σ_i (with $i = H, K, L$) are *positively* correlated. As such, a sign of $\partial \mathcal{G}_i / \partial \sigma_i$ is identical to that of $\partial \mathcal{G}_i / \partial \sigma_g^i$ – what the model tells is consistent with the correlation pattern shown in Ch. 1. So, uncertainty precisely represents the standard deviation.

When $\eta_{HL} > 0$ and/or $\eta_{KL} > 0$, we have the following inequalities:

$$
\frac{\partial \sigma_g^H}{\partial \sigma_H} \gtreqless 0 \Leftrightarrow \sigma_H \gtreqless \eta_{HL}\sigma_L, \quad \frac{\partial \sigma_g^H}{\partial \sigma_L} \gtreqless 0 \Leftrightarrow \sigma_L \gtreqless \eta_{HL}\sigma_H,
$$

$$
\frac{\partial \sigma_g^K}{\partial \sigma_K} \gtreqless 0 \Leftrightarrow \sigma_K \gtreqless \eta_{KL}\sigma_L, \quad \frac{\partial \sigma_g^K}{\partial \sigma_L} \gtreqless 0 \Leftrightarrow \sigma_L \gtreqless \eta_{KL}\sigma_K.
$$

5.C Demographic Uncertainty with Two Correlations

This Appendix has four diagrams (Figs. 5.22 to 5.25) that show the impacts of demographic uncertainty σ_L when *both* correlation parameters are not zero ($\eta_{HL} \neq 0$ *and* $\eta_{KL} \neq 0$). They are self-explanatory.

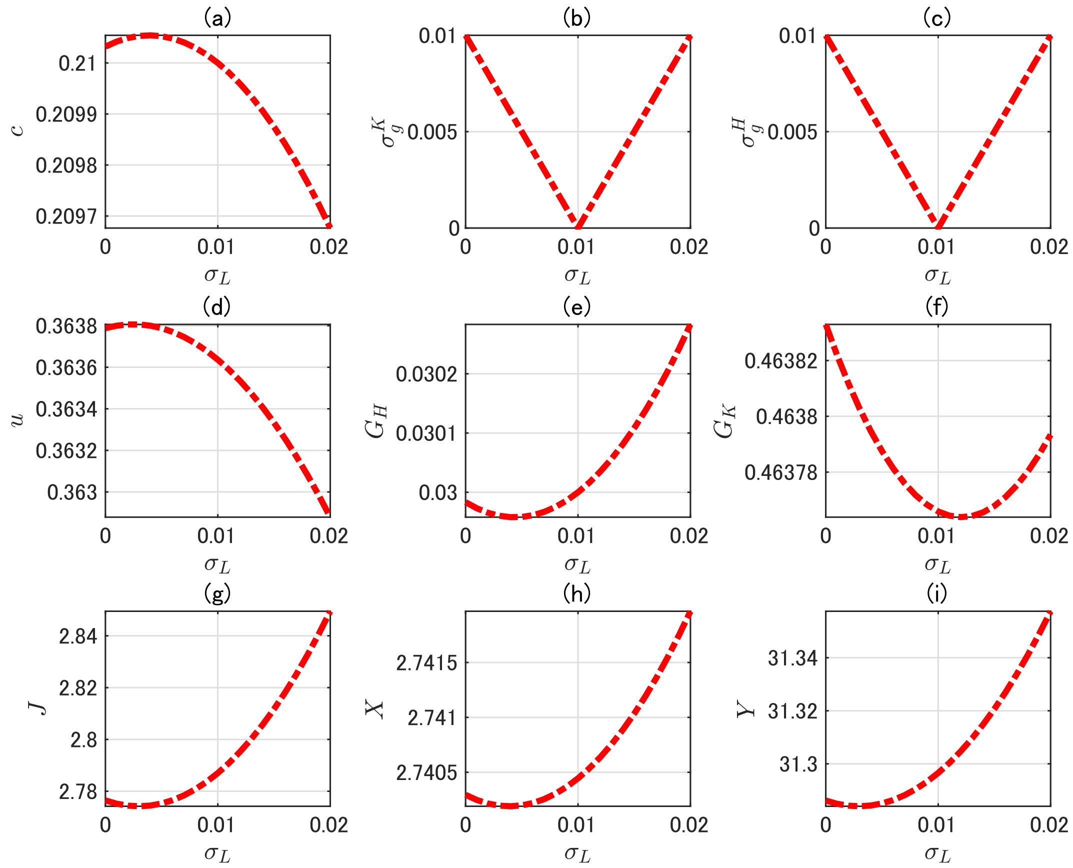

Figure 5.22 The impacts of demographic uncertainty σ_L when $\eta_{HL} > 0$ and $\eta_{KL} > 0$.

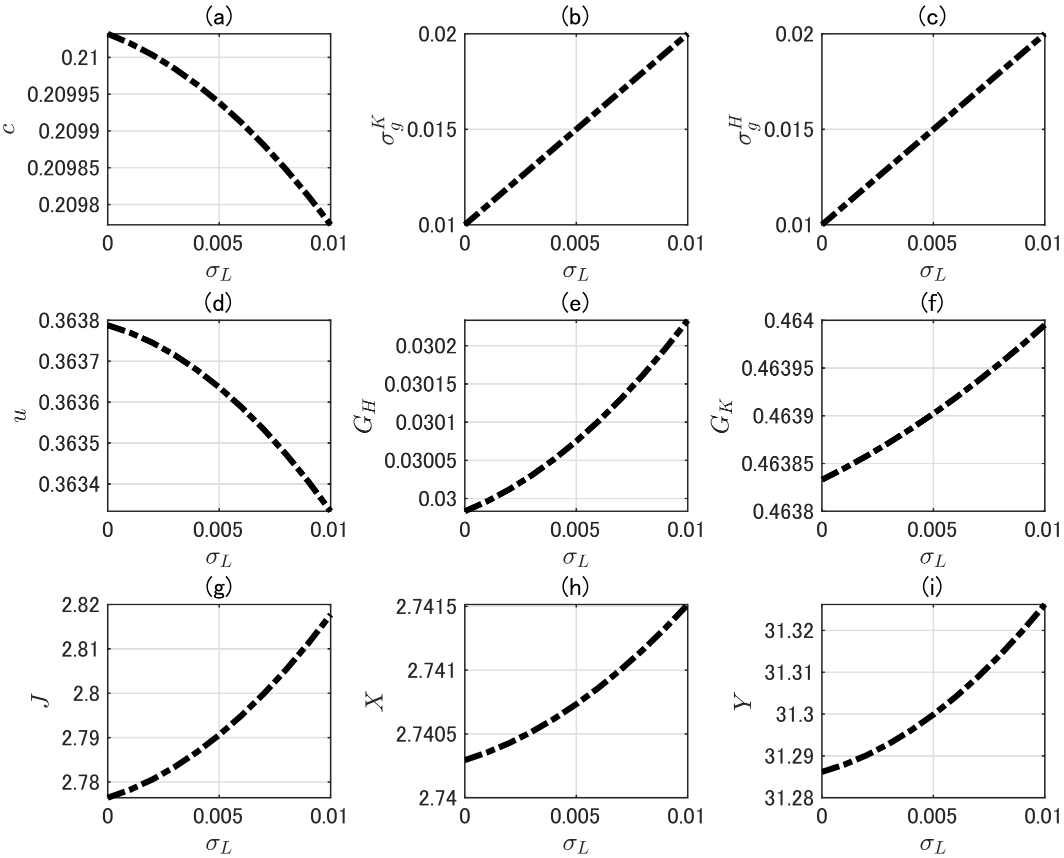

Figure 5.23 The impacts of demographic uncertainty σ_L when $\eta_{HL} < 0$ and $\eta_{KL} < 0$.

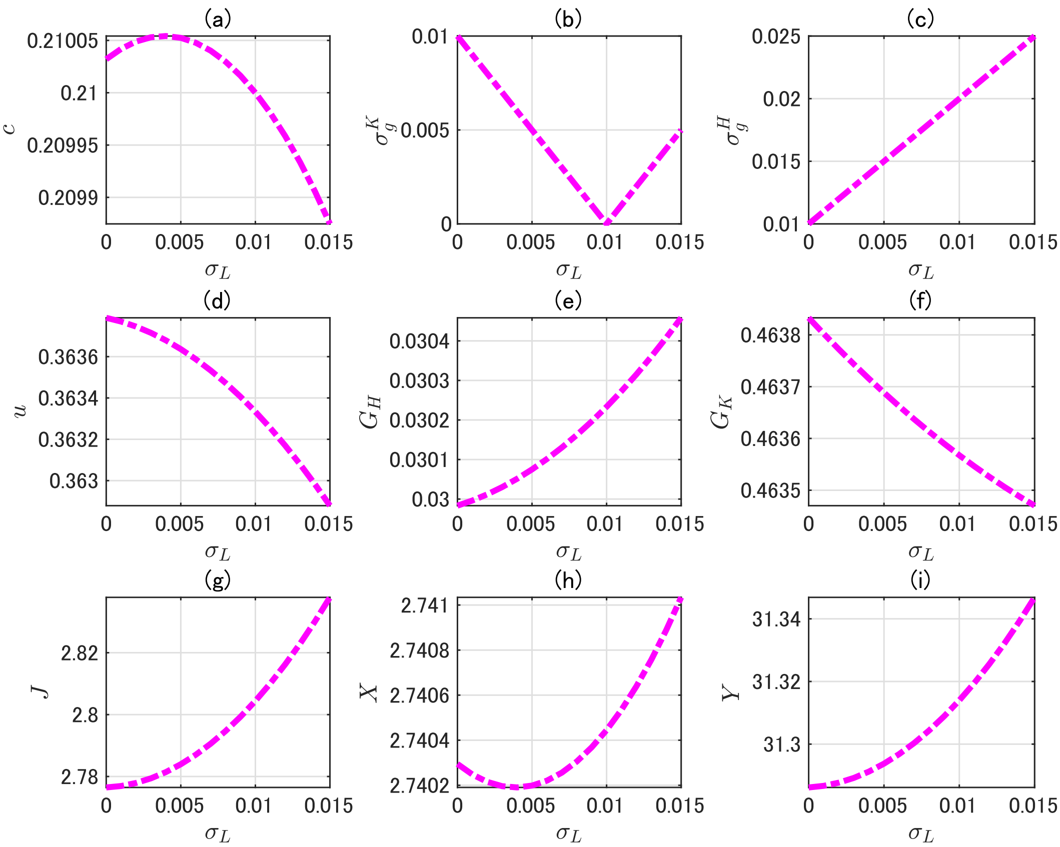

Figure 5.24 The impacts of demographic uncertainty σ_L when $\eta_{HL} < 0$ and $\eta_{KL} > 0$.

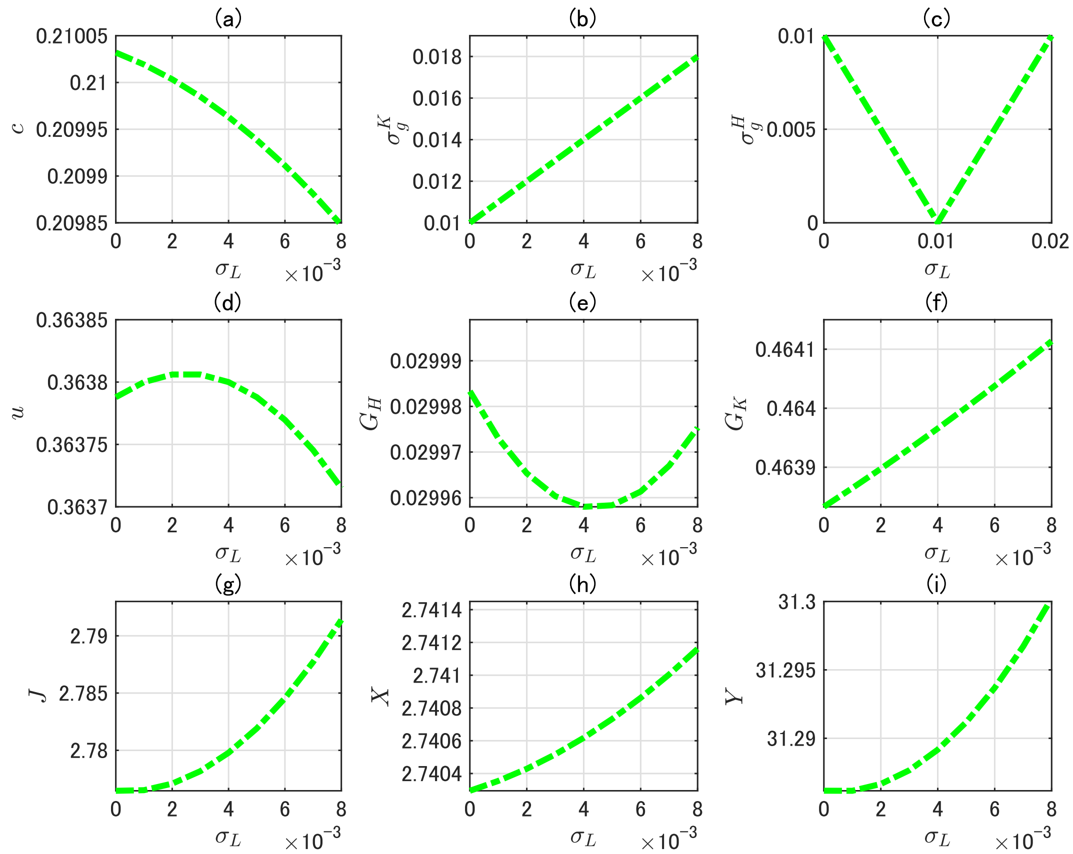

Figure 5.25 The impacts of demographic uncertainty σ_L when $\eta_{HL} > 0$ and $\eta_{KL} < 0$.

Chapter 6

Conclusions

This thesis has analytically explored the relationship between growth and uncertainty. Using the simple Uzawa-Lucas model featuring various types of uncertainty and their correlation, it has successfully replicated all patterns – the negative, positive, U-shaped, and inverted U-shaped. If my theoretical findings are empirically true, as I argue in Ch. 5, policymakers can decide whether to completely wipe out uncertainty or accept small degree of uncertainty to maximize economic growth and welfare.

This thesis has some limitations. First, to solve the model in closed form, I have imposed a Xie (1991) condition throughout. Though analytical solutions make things transparent, the generality of my findings is uncertain; when an economy is far from a steady state where that condition is not satisfied, will my results qualitatively and quantitatively (un)changed? For that purpose, I must numerically evaluate my results. Recently, a continuous-time approach is getting gradually more dominant than a discrete-time approach (for example, see Achdou et al., 2017). Therefore, I expect that numerical solution of HJB equations will be much easier in the near future.

Second, my analysis is not explicit about the time horizon. For example, in Ramey and Ramey (1995), policies that stabilize business cycle fluctuations will increase the growth rate of an economy. The conventional view on macroeconomics, however, is that business cycle fluctuations are short-run phenomena, while economic growth is long-run phenomena. Therefore, in some sense, my model is like RBC: short-run and long-run phenomena are analyzed by the same framework. Therefore, to be explicit about the time horizon, I may need to include essential ingredients of short-run macroeconomics, such as sticky prices.

Third, related the final sentence of the second point, my model is frictionless. For example, it ignores human capital externalities studies in Lucas (1988) and Benhabib and Perli (1994). Without distortions, the central planner problem and the decentralized problem are identical. Thus, there is no room for policy − monetary or fiscal − in my models. Even in the presence of uncertainty, second welfare theorem still holds. Thus, as in Posch (2011) and Posch and Wälde (2011), it would be desirable to explore the role of policy in the decentralized market with some distortion under uncertainty.

Those being said, I believe that this thesis has provided a useful benchmark to investigate these further issues.

Appendix A

The Deterministic Uzawa-Lucas Model

This Appendix reviews a standard Uzawa-Lucas model. By "standard," I mean the model with *no* externality and *no* uncertainty. Thus, it serves for a useful benchmark if you are new to the Uzawa-Lucas model or if you wish to refresh your memory. This Appendix is completely self-contained, so you can understand the main text without reading it.

This Appendix is designed for first-year graduate courses or advanced undergraduate programs in macroeconomics. Though I heavily draw on Benhabib and Perli (BP, 1994) and slightly on Barro and Sala-i-Martin (BS, 2004), I have made it accessible by, for example, providing a step-by-step derivation, making implicit assumptions explicit, using figures, etc.

Table A.1: Ingredients of Three Studies.

Study	**BP (1994)**	**BS (2004)**	**This Appendix**
Externality	○		
Depreciation		○	○
Population			○
$\delta_K = \delta_H$?		Yes	No
Stability	Local Stability	Phase Diagram	Both

Compared with other endogenous growth models such as Romer (1990), the Uzawa-Lucas model is somehow not widely covered in leading textbooks.[1] Therefore, I have written this Appendix as a "textbook" account of the model of Uzawa (1965) and Lucas (1988). As summarized in Tab. A.1, I have simplified but largely extended BP (1994) and BS (2004) by ignoring human capital externalities, con-

[1] For example, this model is put in an exercise in Acemoglu (2009, p.407).

sidering population growth, adding separate depreciation rates of capital ($\delta_K \neq \delta_H$), and undertaking the stability analysis of a steady state both with a local stability analysis and a phase diagram approach. As such, I hope that even a reader familiar with the Uzawa-Lucas model will discover new insights in a place or two; I have indeed learned a tremendous amount in preparing this Appendix.

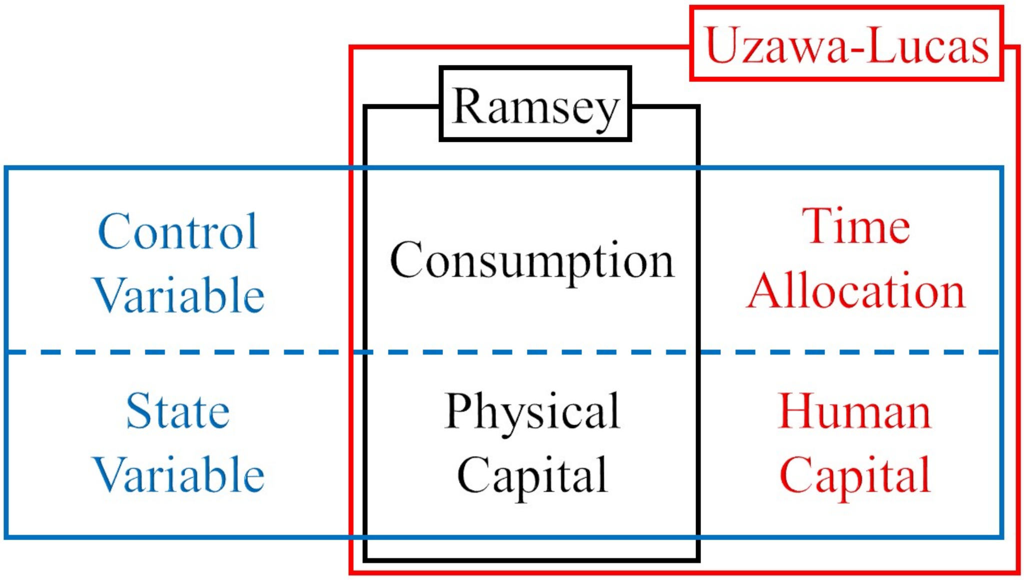

Figure A.1 Ramsey and Uzawa-Lucas model: comparison.

As a prelude, what makes the Uzawa-Lucas model difficult is the number of major variables in the model. In the cornerstone model of modern macroeconomics of the Ramsey, as summarized in Fig. A.1, there is only *one* control variable (consumption) and *one* state variable (physical capital). In contrast, the Uzawa-Lucas model has *two* control variables (consumption plus *time allocation*) and *two* state variables (physical and *human* capital). As a result, it will consist of the *four*-dimensional system. Carefully reading this Appendix, however, you can solve and understand the main mechanism of the Uzawa-Lucas model.

A.1 The Textbook Model

Consider a closed economy in continuous time running to an infinite horizon. It consists of a set of identical households (with measure normalized to 1). Population within each household grows at the rate n. All members of the household supply their one unit of labor inelastically.

The household is fully altruistic toward all of its future members and always makes the allocations of consumption among household members cooperatively. The utility function of each household at time $t = 0$ is

$$\int_0^\infty e^{-\rho t} \left(\frac{c(t)^{1-\phi} - 1}{1 - \phi} \right) N(t) dt,$$

where $c(t)$ is consumption per capita at time t. $\rho > 0$ is the subjective discount rate. $\phi > 0$ is the coefficient of relative risk aversion.[1] $N(t)$ is the size of the representative household (equal to total population, since the measure of households is normalized to 1).

Let there be $N_H(t)$ workers with skill level $H \in [0, \infty]$, so that

$$N(t) = \int_0^\infty N_H(t) dH.$$

By an individual's human capital, Lucas (1988, p.17) means its general skill level; a worker with human capital $H(t)$ is the productive equivalent of two workers with $\frac{1}{2}H(t)$ each, or a half-time worker with $2H(t)$. The theory of human capital focuses on the fact that the way an individual allocates its time over various activities in the current period affects its productivity, or its $H(t)$ level, in future periods.

Suppose a worker with skill H devotes the fraction $u(H)$ of its non-leisure time to current production, and the remaining $1 - u(H)$ to human capital accumulation, as illustrated in Fig. A.2. Then, the effective workforce in production is the sum

$$N^e(t) = \int_0^\infty u(H) N(H) H dH,$$

of the skill-weighted hours devoted to current production. If all workers have skill level $H(t)$ and all choose the time allocation $u(t)$, the effective workforce is just $N^e(t) = u(t)H(t)N(t)$.

[1] The inverse $1/\phi$ is called the intertemporal elasticity of substitution. It measures how easily the household can substitute consumption at different points of time. With time separable utility functions, the inverse of the elasticity of intertemporal substitution and the coefficient of relative risk aversion are identical. Therefore, the family of constant relative risk aversion (CRRA) utility functions also consists of those functions with constant elasticity of intertemporal substitution (Acemoglu, 2009, p.308). When $\phi > 1$, consumption at different times are poor substitutes for one another; as $\phi \to \infty$, households become infinitely risk-averse and infinitely unwilling to substitute consumption over time. When $\phi \in (0, 1)$, the elasticity of substitution is bigger than 1 and the household finds it much easier to trade consumption now for consumption later.

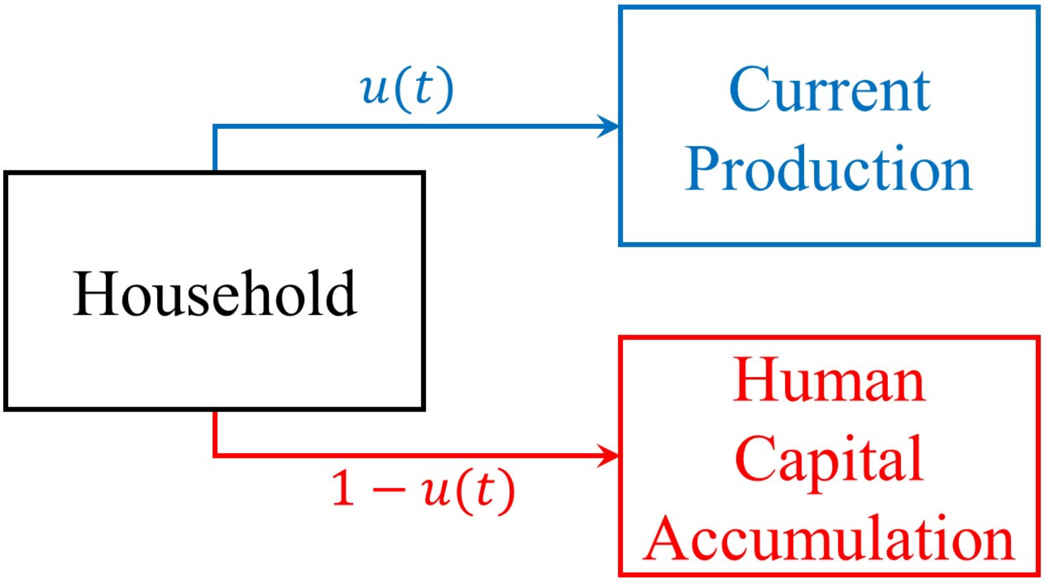

Figure A.2 Uzawa-Lucas model: a conceptual framework.

Production per capita of the one good is divided into consumption $c(t)$ and physical capital accumulation. Let $K(t)$ denote the total stock of physical capital and $\dot{K}(t) \equiv dk(t)/dt$ its rate of change. Then, the physical capital accumulation equation is

$$\dot{K}(t) = \underbrace{AK(t)^{\beta}[u(t)H(t)N(t)]^{1-\beta}}_{\equiv Y(t)} - c(t)N(t) - \delta_K K(t), \tag{A.1}$$

where $Y(t)$ is output. A is a constant that represents the technology level. $\beta \in (0,1)$ is the physical capital-share parameter. $\delta_K \in (0,1)$ is the depreciation rate of physical capital.

To complete the model, the effort $1-u(t)$ devoted to the accumulation of human capital must be linked to the rate of change in its level, $H(t)$. As in Uzawa (1965) and Lucas (1988), a technology relating the growth of human capital, $\dot{H}(t) \equiv dH(t)/dt$, to the level already attained and the effort devoted to acquiring more is

$$\dot{H}(t) = b(1-u(t))H(t) - \delta_H H(t), \tag{A.2}$$

where $b > 0$ indicates the efficiency of human capital accumulation.[1] $\delta_H \in (0,1)$ captures its depreciation rate, which comes about, for example, because new ma-

[1]Kuwahara (2017) endogenizes b and studies the indeterminacy of equilibria, where different countries follow different equilibrium trajectories toward a balanced growth path.

chines and techniques are introduced that erode the existing human capital of the worker (Uzawa (1965) and Lucas (1988) assume $\delta_H = 0$). According to this equation, if no effort is devoted to human capital accumulation ($u(t) = 1$), then none accumulates. If all effort is devoted to this purpose ($u(t) = 0$), $H(t)$ grows at its maximal rate b (net of δ_H).

Before carrying on, note that $\dot{H}(t)$ is *linear* in $H(t)$. On this linearity assumption, Lucas (1988, p.19) explains as follows: "...we seem to see diminishing returns in observed, individual patterns of human capital accumulation: people accumulate it rapidly early in life, then less rapidly, then not at all − as though each additional percentage increment were harder to gain than the preceding one. But an alternative explanation for *this* observation is simply that an individual's lifetime is finite, so that the return to increments falls with time." See Jones (1995; 2005; 2019) for a more in-depth discussion of the *linearity critique* in many endogenous growth models.

We can now do the optimization in continuous time. Let $\mathcal{H}(t)$ denote the current value Hamiltonian that reflects the utility value of what gets produced at time t; it is the utility equivalent of net domestic product. We maximize $\mathcal{H}(t)$ instantaneously with respect to two control variables $c(t)$ and $u(t)$. The Hamiltonian is

$$\mathcal{H}(t) = \left(\frac{c(t)^{1-\phi}}{1-\phi} - \frac{1}{1-\phi}\right) N(t) + \theta_1(t) \underbrace{\left(AK(t)^{\beta}[u(t)H(t)N(t)]^{1-\beta} - c(t)N(t) - \delta_K K(t)\right)}_{=\dot{K}(t)}$$
$$+ \theta_2(t) \underbrace{\left(b(1-u(t))H - \delta_H H(t)\right)}_{=\dot{H}(t)},$$

where $\theta_1(t)$ is the shadow price or co-state variable for physical capital $K(t)$. $\theta_2(t)$ is the shadow price of human capital $H(t)$. The co-state variables estimate the marginal value of the associated state variables $K(t)$ and $H(t)$.

The Pontryagin's maximum principle implies, first,

$$\frac{\partial \mathcal{H}(t)}{\partial c(t)} = 0 \Leftrightarrow c(t)^{-\phi} = \theta_1(t), \tag{A.3}$$

and

$$\frac{\partial \mathcal{H}(t)}{\partial u(t)} = 0 \Leftrightarrow (1-\beta)AK(t)^{\beta}(u(t)H(t)N(t))^{-\beta}\theta_1(t)N(t) = \theta_2(t)b. \tag{A.4}$$

Eq. (A.3) says that the marginal utility of consumption $c(t)^{-\phi}$ must equal, at each instant, the value of the marginal utility of net investment $\theta_1(t)$. Eq. (A.4) says that the value of marginal unit of time devoted to study must equal the value of the marginal unit of time devoted to production.

Next, we have two co-state equations:

$$\dot{\theta}_1(t) = \rho\theta_1 - \frac{\partial \mathcal{H}(t)}{\partial K(t)} \Leftrightarrow \frac{\dot{\theta}_1(t)}{\theta_1(t)} = \rho - \underbrace{\beta AK(t)^{\beta-1}\left(u(t)H(t)N(t)\right)^{1-\beta}}_{\text{marginal product of physical capital}} + \delta_K, \tag{A.5}$$

$$\begin{aligned}\dot{\theta}_2(t) &= \rho\theta_2 - \frac{\partial \mathcal{H}(t)}{\partial H(t)} \\ &\Leftrightarrow \frac{\dot{\theta}_2(t)}{\theta_2(t)} = \rho - \frac{\theta_1(t)}{\theta_2(t)}\underbrace{(1-\beta)AK(t)^{\beta}H(t)^{-\beta}(u(t)N(t))^{1-\beta}}_{\text{marginal product of human capital}} - b(1-u(t)) + \delta_H.\end{aligned} \tag{A.6}$$

Finally, we have the transversality condition (TVC):

$$\lim_{t\to\infty} e^{-\rho t}[\theta_1(t)K(t) + \theta_2(t)H(t)] = 0. \tag{A.7}$$

Eq. (A.7) assets that the value of physical and human capital must approach zero as time approaches ∞; otherwise, there would be a tendency to postpone consumption forever – optimizing agents must not have any valuable "assets" left over at the *end* of the planning horizon.

From Eqs. (A.3) and (A.5), the Euler equation of the representative household is obtained as

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\phi}\left(\underbrace{\beta AK(t)^{\beta-1}\left(u(t)H(t)N(t)\right)^{1-\beta}}_{\text{marginal product of physical capital}} - \rho - \delta_K\right). \tag{A.8}$$

Thus, consumption will be increasing or decreasing according to whether the marginal product of physical capital is greater or less than the rate of time preference (net of δ_K). In the former case, the household is relatively patient and finds it optimal to reduce consumption in the short run, allowing it to increase over time.

From Eqs. (A.3) and (A.4), we have

$$(1-\beta)AK(t)^{\beta}(u(t)H(t)N(t))^{-\beta}c(t)^{-\phi}N(t) = \theta_2(t)b.$$

Taking derivatives of both sides with respect to time, we get

$$\frac{\dot{\theta}_2(t)}{\theta_2(t)} = \rho - \beta\frac{c(t)N(t)}{K(t)} - \beta\frac{\dot{u}(t)}{u(t)} - \beta b(1-u(t)) + \beta\delta_H + (1-\beta)(n+\delta_K). \quad \text{(A.9)}$$

Using Eqs. (A.6) and (A.9), we find

$$\begin{aligned}&\rho - \frac{\theta_1(t)}{\theta_2(t)}(1-\beta)A\left(\frac{K(t)}{H(t)}\right)^{\beta}(u(t)N(t))^{1-\beta} - b(1-u(t)) + \delta_H \\ &= -\beta\frac{c(t)N(t)}{k(t)} - \beta\frac{\dot{u}(t)}{u(t)} - \beta b(1-u(t)) + \beta\delta_H + \rho + (1-\beta)(n+\delta_K).\end{aligned}$$

To solve this, note from Eq. (A.4) that the ratio of shadow prices is

$$\frac{\theta_1(t)}{\theta_2(t)} = \frac{bu(t)^{1-\beta}}{(1-\beta)A\left(\frac{K(t)}{H(t)}\right)^{\beta}N^{1-\beta}}$$

Inserting this ratio into the equation above, after some algebra, we find the law of motion for $u(t)$:

$$\frac{\dot{u}(t)}{u(t)} = bu(t) - \frac{c(t)N(t)}{K(t)} + \left(\frac{1-\beta}{\beta}\right)(b+n+\delta_K-\delta_H). \quad \text{(A.10)}$$

To recap, we have the four-dimensional system in $K(t)$, $H(t)$, $c(t)$, and $u(t)$ that consists of Eqs. (A.1), (A.2), (A.8), and (A.10).

A.2 The Reduced Model

We next reduce the system by one dimension, as the analytical study of a three-dimensional system is much simpler. We can do this by a change of variables; that is, by defining the following new stationary variables:

$$\chi(t) \equiv \frac{K(t)}{H(t)N(t)}, \quad q(t) \equiv \frac{c(t)N(t)}{K(t)}.$$

In this way, we can get the law of motion for $\chi(t)$:

$$\begin{aligned}\frac{\dot{\chi}(t)}{\chi(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{H}(t)}{H(t)} - \frac{\dot{N}(t)}{N(t)} \\ &= AK(t)^{\beta-1}u(t)^{1-\beta}(H(t)N(t))^{1-\beta} - \frac{c(t)N(t)}{K(t)} - \delta_K - b(1-u(t)) + \delta_H - n \\ &= A\chi(t)^{\beta-1}u(t)^{1-\beta} - q(t) - \delta_K - b(1-u(t)) + \delta_H - n.\end{aligned} \tag{A.11}$$

By the same token, the law of motion for $q(t)$ is:

$$\begin{aligned}\frac{\dot{q}(t)}{q(t)} &= \frac{\dot{c}(t)}{c(t)} + \frac{\dot{N}(t)}{N(t)} - \frac{\dot{K}(t)}{K(t)} \\ &= \frac{A\beta}{\phi}K(t)^{\beta-1}(u(t)H(t)N(t))^{1-\beta} - \frac{\rho+\delta_K}{\phi} + n - AK(t)^{\beta}(u(t)H(t)N(t))^{1-\beta} \\ &+ \frac{c(t)N(t)}{K(t)} + \delta_K \\ &= \frac{A\beta}{\phi}A\chi(t)^{\beta-1}u(t)^{1-\beta} - A\chi(t)^{\beta-1}u(t)^{1-\beta} + q(t) - \frac{\rho}{\phi} - \frac{\delta_K}{\phi} + \delta_K + n \\ &= A\left(\frac{\beta-\phi}{\phi}\right)\chi(t)^{\beta-1}u(t)^{1-\beta} + q(t) + \frac{\phi n - \rho + \delta_K(\phi-1)}{\phi}.\end{aligned} \tag{A.12}$$

You can see that $K(t)$, $H(t)$, and $c(t)$ don't appear in Eqs. (A.11) and (A.12). So, we successfully have a new system in only three dimensions, $\chi(t)$, $q(t)$, and $u(t)$. Together with Eq. (A.10), our three-dimensional system is

$$\dot{\chi}(t) = A\chi(t)^{\beta}u^{1-\beta} - b(1-u(t))\chi(t) - (n+\delta_K-\delta_H)\chi(t) - q(t)\chi(t), \tag{A.13}$$

$$\dot{q}(t) = A\left(\frac{\beta-\phi}{\phi}\right)\left(\frac{u(t)}{\chi(t)}\right)^{1-\beta} q(t) + q(t)^2 + \frac{\phi n - \rho + \delta_K(\phi-1)}{\phi}q(t), \tag{A.14}$$

$$\dot{u}(t) = bu(t)^2 - q(t)u(t) + \left(\frac{1-\beta}{\beta}\right)(b + n + \delta_K - \delta_H)u(t). \tag{A.15}$$

Our next task is to find the steady states of this reduced system.

A.3 Steady State

From Eq. (A.15), the steady state value of $q(t)$ is a function of $u(t)$:

$$\begin{aligned} 0 &= bu^{*2} - q^*u^* + \left((b + n + \delta_K - \delta_H)\left(\frac{1-\beta}{\beta}\right)\right)u^* \\ \Leftrightarrow q^*u^* &= bu^{*2} + \left((b + n + \delta_K - \delta_H)\left(\frac{1-\beta}{\beta}\right)\right)u^* \\ \Leftrightarrow q^* &= bu^* + \left((b + n + \delta_K - \delta_H)\left(\frac{1-\beta}{\beta}\right)\right). \end{aligned} \tag{A.16}$$

Next, from Eq. (A.14), the steady state value of $\chi(t)$ is again a function of $u(t)$:

$$\begin{aligned} 0 &= A\left(\frac{\beta-\phi}{\phi}\right)\chi^{*\beta-1}u^{*1-\beta}q^* + q^{*2} + \left(\frac{\phi n - \rho + \delta_K(\phi-1)}{\phi}\right)q^* \\ \Leftrightarrow A\left(\frac{\beta-\phi}{\phi}\right)\chi^{*\beta-1}u^{*1-\beta} &= -q^* - \left(\frac{\phi n - \rho + \delta_K(\phi-1)}{\phi}\right) \\ \Leftrightarrow A(\beta-\phi)\chi^{*\beta-1}u^{*1-\beta} &= -b\phi u^* - \phi(b + n - \delta_H + \delta_K)\left(\frac{1-\beta}{\beta}\right) + \rho - \delta_K(\phi-1) - n\phi \\ \Leftrightarrow \chi^* &= \left(\frac{A(\beta-\phi)}{\rho - \phi\left((b + n - \delta_H + \delta_K)\left(\frac{1-\beta}{\beta}\right)\right) - \delta_K(\phi-1) - b\phi u^* - n\phi}\right)^{\frac{1}{1-\beta}} u^* \end{aligned} \tag{A.17}$$

Finally, from Eq. (A.13), we can calculate the steady state value of $u(t)$ as a function of the parameters only:

$$
\begin{aligned}
0 &= A\chi^{*\beta}u^{*1-\beta} - b(1-u^*)\chi^* - (n+\delta_K-\delta_H)\chi^* - q^*\chi^* \\
&\Leftrightarrow A\chi^{*\beta}u^{*1-\beta} = (b+n+\delta_K-\delta_H)\left(1+\frac{1-\beta}{\beta}\right) \\
&\Leftrightarrow \chi^{*\beta}u^{*1-\beta} = \frac{b+n+\delta_K-\delta_H}{A\beta} \\
&\Leftrightarrow \frac{-b\phi u^* - \phi(b+n-\delta_H+\delta_K)\left(\frac{1-\beta}{\beta}\right) + \rho - \delta_K(\phi-1) - n\phi}{A(\beta-\phi)} = \frac{b+n+\delta_K-\delta_H}{A\beta} \\
&\Leftrightarrow -b\phi\beta u^* = -\beta\rho + \beta\delta_K(\phi-1) + \beta n\phi + (b+n+\delta_K-\delta_H)(1-\phi)\beta \\
&\Leftrightarrow u^* = \frac{b(\phi-1)+\rho-n+(1-\phi)\delta_H}{b\phi},
\end{aligned}
\tag{A.18}
$$

where we require $u^* \in (0,1)$. For $u^* > 0$, we need

$$\rho > n + (1-\phi)(b-\delta_H).$$

When $\phi = 1$, this boils down to $\rho > n$: the condition that ensures the discounting of future utility streams (otherwise, the utility function would have inifinite value). On the other hand, the condition for $u^* < 1$ is

$$\rho < b + n - (1-\phi)\delta_H.$$

When $\phi = 1$ and $n = 0$, this boils down to $b > \rho$: the condition for positive long-run growth in the Uzawa-Lucas model.

Taking stock, Eqs. (A.16), (A.17), and (A.18) represent the steady state of the reduced model. Now, we must check whether this steady state satisfies the TVC (A.7).

A.4 Transversality Condition

For the first condition, we must have

$$\lim_{t\to\infty}\left(-\rho + \frac{\dot{\theta}_1(t)}{\theta_1(t)} + \frac{\dot{K}(t)}{K(t)}\right) < 0.$$

From Eqs. (A.1) and (A.5), the first TVC reduces to

$$
\begin{aligned}
&\left((1-\beta)AK^{*\beta-1}(u^*H^*N^*)^{1-\beta}-q^*\right)<0\\
&\Leftrightarrow -\left(-(1-\beta)A\left(\frac{u^*}{\chi^*}\right)^{1-\beta}+bu^*+(b+n+\delta_K-\delta_H)\left(\frac{1-\beta}{\beta}\right)\right)<0\\
&\Leftrightarrow -bu^*<0.
\end{aligned}
$$

Because $b>0$ and $u^*\in(0,1)$, the first TVC is always satisfied. For the second TVC, we need

$$
\lim_{t\to\infty}\left(-\rho+\frac{\dot{\theta}_2(t)}{\theta_2(t)}+\frac{\dot{H}(t)}{H(t)}\right)<0.
$$

From Eqs. (A.2) and (A.6), the second TVC reduces to

$$
\begin{aligned}
&\left(-\rho+\rho-\frac{\theta_1}{\theta_2}(1-\beta)AK^{*\beta}H^{*-\beta}u^{*1-\beta}N^{*1-\beta}-b(1-u^*)+\delta_H+b(1-u^*)-\delta_H\right)<0\\
&\Leftrightarrow -bu^*<0.
\end{aligned}
$$

So, the second TVC is always satisfied as well. As the Hamiltonian is concave, and as both the TVCs are satisfied at the steady state, the solution described above is a maximum.

A.5 Stability Analysis I

Our final task is to investigate the local stability properties of the steady state. I first use a Routh-Hurwitz stability criterion.[1] I will construct a phase diagram in the next section.

To set up the Jacobian, we calculate the first-order partial derivatives of our three-dimensional system that consists of Eqs. (A.10), (A.11), and (A.12) as follows:

$$
J_{11}\equiv\frac{\partial\dot{\chi}(t)}{\partial\chi(t)}=\beta A\chi(t)^{\beta-1}u(t)^{1-\beta}-b(1-u(t))-(n+\delta_K-\delta_H)-q(t),
$$

[1] See Benhabib and Perli (1994, p.139) for an exposition of this theorem. Kuwahara (2017) applies it to the Uzawa-Lucas model, and Kuwahara (2019) to the Romer (1990) model.

$$J_{12} \equiv \frac{\partial \dot{\chi}(t)}{\partial u(t)} = (1-\beta)A\chi(t)^{\beta}u(t)^{-\beta} + b\chi(t), \quad J_{13} \equiv \frac{\partial \dot{\chi}(t)}{\partial q(t)} = -\chi(t),$$

$$J_{21} \equiv \frac{\partial \dot{u}(t)}{\partial \chi(t)} = 0, \quad J_{22} \equiv \frac{\partial \dot{u}(t)}{\partial u(t)} = 2bu(t) - q(t) + \left((b+n+\delta_K-\delta_H)\left(\frac{1-\beta}{\beta}\right)\right),$$

$$J_{23} \equiv \frac{\partial \dot{\chi}(t)}{\partial q(t)} = -u(t), \quad J_{31} \equiv \frac{\partial \dot{q}(t)}{\partial \chi(t)} = A\left(\frac{\beta-\phi}{\phi}\right)(\beta-1)\chi(t)^{\beta-2}u(t)^{1-\beta}q(t),$$

$$J_{32} \equiv \frac{\partial \dot{q}(t)}{\partial u(t)} = A\left(\frac{\beta-\phi}{\phi}\right)(1-\beta)\chi(t)^{\beta-1}u(t)^{-\beta}q(t),$$

$$J_{33} \equiv \frac{\partial \dot{q}(t)}{\partial q(t)} = A\left(\frac{\beta-\phi}{\phi}\right)\left(\frac{u(t)}{\chi(t)}\right)^{1-\beta} + 2q(t) + \frac{\phi n - \rho + \delta_K(\phi-1)}{\phi}.$$

To obtain the Jacobian evaluated at the steady state, for example, consider J_{11} (remember Eq. A.11):

$$\begin{aligned} J_{11} &= \frac{\dot{\chi}(t)}{\chi(t)} - A\chi(t)^{\beta-1}u(t)^{1-\beta} + \beta A\chi(t)^{\beta-1}u(t)^{1-\beta} \\ &= \frac{\dot{\chi}(t)}{\chi(t)} - (1-\beta)A\left(\frac{u(t)}{\chi(t)}\right)^{1-\beta}. \end{aligned}$$

Therefore, J_{11}^* – the value of J_{11} at the steady state (SS) – is

$$J_{11}^* \equiv \left.\frac{\partial \dot{\chi}(t)}{\partial \chi(t)}\right|_{SS} = -(1-\beta)A\left(\frac{u^*}{\chi^*}\right)^{1-\beta}.$$

Using this technique, we can calculate the rest:

$$J_{12}^* \equiv \left.\frac{\partial \dot{\chi}(t)}{\partial u(t)}\right|_{SS} = -\frac{\chi^*}{u^*}(J_{11}^* - bu^*), \quad J_{13}^* \equiv \left.\frac{\partial \dot{\chi}(t)}{\partial u(t)}\right|_{SS} = -\chi^*,$$

$$J_{21}^* \equiv \left.\frac{\partial \dot{u}(t)}{\partial \chi(t)}\right|_{SS} = 0, \quad J_{22}^* \equiv \left.\frac{\partial \dot{u}(t)}{\partial u(t)}\right|_{SS} = bu^*, \quad J_{23}^* \equiv \left.\frac{\partial \dot{u}(t)}{\partial q(t)}\right|_{SS} = -u^*,$$

$$J_{31}^* \equiv \left.\frac{\partial \dot{q}(t)}{\partial \chi(t)}\right|_{SS} = J_{11}^* \left(\frac{\beta - \phi}{\phi}\right) \frac{q^*}{\chi^*}, \quad J_{32}^* \equiv \left.\frac{\partial \dot{q}(t)}{\partial u(t)}\right|_{SS} = -J_{11}^* \left(\frac{\beta - \phi}{\phi}\right) \frac{q^*}{\chi^*},$$

$$J_{33}^* \equiv \left.\frac{\partial \dot{q}(t)}{\partial q(t)}\right|_{SS} = q^*.$$

So, the Jacobian matrix evaluated at the steady state J^* is

$$J^* = \begin{pmatrix} J_{11}^* & -\frac{\chi^*}{u^*}(J_{11}^* - bu^*) & -\chi^* \\ 0 & bu^* & -u^* \\ J_{11}^* \left(\frac{\beta-\phi}{\phi}\right) \frac{q^*}{\chi^*} & -J_{11}^* \left(\frac{\beta-\phi}{\phi}\right) \frac{q^*}{u^*} & q^* \end{pmatrix}. \tag{A.19}$$

Let TrJ^* denote a trace of J^* and $DetJ^*$ a determinant of J^*. Then, the eigenvalues of J^* are the solution of its characteristic equation

$$-\Lambda^3 + TrJ^*\Lambda^2 - J_B^*\Lambda + DetJ^* = 0, \tag{A.20}$$

where

$$TrJ^* = J_{11}^* + q^* + bu^* = 2bu^* > 0,$$

$$\begin{aligned}
J_B^* &= \begin{vmatrix} J_{11}^* & -\frac{\chi^*}{u^*}(J_{11}^* - bu^*) \\ 0 & bu^* \end{vmatrix} + \begin{vmatrix} bu^* & -u^* \\ -J_{11}^* \left(\frac{\beta-\phi}{\phi}\right) \frac{q^*}{u^*} & q^* \end{vmatrix} + \begin{vmatrix} J_{11}^* & -\chi^* \\ J_{11}^* \left(\frac{\beta-\phi}{\phi}\right) \frac{q^*}{\chi^*} & q^* \end{vmatrix} \\
&= J_{11}^* bu^* + q^* bu^* - J_{11}^* \left(\frac{\beta - \phi}{\phi}\right) q^* + J_{11}^* q^* + J_{11}^* \left(\frac{\beta - \phi}{\phi}\right) q^* \\
&= J_{11}^* q^* + bu^* q^* + bJ_{11}^* u^* = J_{11}^* q^* + bu^*(J_{11}^* + q^*) \\
&= \underbrace{(bu^* - q^*)q^*}_{(+)} + \underbrace{b^2 u^{*2}}_{(+)} > 0,
\end{aligned}$$

and

$$
\begin{aligned}
DetJ^* &= J_{11}^* \begin{vmatrix} bu^* & -u^* \\ -J_{11}^* & q^* \end{vmatrix} + \frac{\chi^*}{u^*}(J_{11}^* - bu^*) \begin{vmatrix} 0 & -u^* \\ J_{11}^* \left(\frac{\beta-\phi}{\phi}\right)\frac{q^*}{\chi^*} & q^* \end{vmatrix} \\
&- \chi^* \begin{vmatrix} 0 & bu^* \\ J_{11}^* \left(\frac{\beta-\phi}{\phi}\right)\frac{q^*}{\chi^*} & -J_{11}^* \left(\frac{\beta-\phi}{\phi}\right)\frac{q^*}{u^*} \end{vmatrix} \\
&= J_{11}^* \left(bu^*q^* - J_{11}q^* \left(\frac{\beta-\phi}{\phi}\right)\right) + \frac{\chi^*}{u^*}(J_{11}^* - bu^*)\left(0 + J_{11}^*\left(\frac{\beta-\phi}{\phi}\right)\frac{q^*}{\chi^*}u^*\right) \\
&- \chi^* \left(0 - bJ_{11}\left(\frac{\beta-\phi}{\phi}\right)\frac{q^*}{\chi^*}u^*\right) \\
&= J_{11}^* bu^*q^* - J_{11}^{*2}q^*\left(\frac{\beta-\phi}{\phi}\right) + J_{11}^{*2}q^*\left(\frac{\beta-\phi}{\phi}\right) - J_{11}^* bu^*q^*\left(\frac{\beta-\phi}{\phi}\right) \\
&+ J_{11}^* bu^*q^*\left(\frac{\beta-\phi}{\phi}\right) = \underbrace{J_{11}^*}_{(-)}\underbrace{bu^*q^*}_{(+)} < 0.
\end{aligned}
$$

Because the initial condition $\chi(0)$ is given but both $q(0)$ and $u(0)$ are free, the competitive equilibrium solution is locally unique (or the steady state is determinate) *if* the Jacobian of the reduced system has *two* eigenvalues with *positive* real parts and *one* with *negative* real part (see Benhabib and Perli, 1994, p.123). For our purpose, let us use the Routh-Hurwitz stability theorem:

Routh-Hurwitz Theorem. *The number of roots of the polynomial in Eq. (A.20) with positive real parts is equal to the number of variations of sign in the scheme*

$$
-1, \quad TrJ^*, \quad -J_B^* + \frac{DetJ^*}{TrJ^*}, \quad DetJ^*.
$$

Proof. See Benhabib and Perli (1994, p.139).

For example, if $TrJ^* > 0$, $-J_B^* + DetJ^*/TrJ^* < 0$, and $DetJ^* > 0$, this scheme is characterized as $\{- + - +\}$. As there are *three* variations of sign (from $-$ to $+$, $+$ to $-$, and $-$ to $+$), Eq. (A.20) has *three* roots with positive real parts in this example. Therefore, a Routh-Hurwitz theorem helps us know the number of roots with positive (negative) real parts without explicitly solving a complex characteristic equation.

In our case, we have $TrJ^* > 0$, $-J_B^* + DetJ^*/TrJ^* < 0$, and $DetJ^* < 0$. So, our scheme is characterized as $\{- + - -\}$. Since there are *two* variations of sign (from $-$ to $+$, and then $+$ to $-$), J^* has *two* roots with positive real parts and *one* root with negative real part; that is, our equilibrium is *locally stable*. This observation is consistent with the Uzawa-Lucas model without human capital externalities. In the presence of externalities, we would have the indeterminacy of equilibria, as in Benhabib and Perli (1994).

A.6 Stability Analysis II

Armed with a Routh-Hurwitz stability theorem, we now know that our dynamic system is locally stable. Constructing a phase diagram, however, we can diagrammatically understand more about the dynamic characteristic of our model. To this end, following Barro and Sala-i-Martin (2004, p.253), let us define the gross average product of physical capital in the production of goods:

$$z(t) \equiv A \left(\frac{u(t)}{\chi(t)} \right)^{1-\beta}.$$

Its steady state value is

$$z^* = A \left(\frac{u^*}{\chi^*} \right)^{1-\beta} = A \times \frac{b+n+\delta_K-\delta_H}{A\beta} = \frac{b+n+\delta_K-\delta_H}{\beta},$$

and its growth rate is

$$\begin{aligned}
\frac{\dot{z}(t)}{z(t)} &= (1-\beta)\left(\frac{\dot{u}(t)}{u(t)} - \frac{\dot{\chi}(t)}{\chi(t)} \right) \\
&= (1-\beta)\left(\left(\frac{1-\beta}{\beta} \right)(b+n+\delta_K-\delta_H) - A\left(\frac{u(t)}{\chi(t)} \right)^{1-\beta} + (b+n+\delta_K-\delta_H) \right) \\
&= (1-\beta)\left(-z(t) + \frac{b+n+\delta_K-\delta_H}{\beta} \right).
\end{aligned} \tag{A.21}$$

We can linearize the three deterministic differential equations (A.21), (A.10), and (A.12). A first-order Taylor expansion around the steady state (z^*, q^*, u^*) gives

$$\frac{\dot{z}(t)}{z(t)} = -(1-\beta)(z(t)-z^*),$$

$$\frac{\dot{q}(t)}{q(t)} = \left(\frac{\beta - \phi}{\phi}\right)(z(t) - z^*) + (q(t) - q^*),$$

$$\frac{\dot{u}(t)}{u(t)} = b(u(t) - u^*) - (q - q^*).$$

From these equations, we can obtain three loci; first, the $\dot{z} = 0$ locus is

$$z = z^*.$$

Second, the $\dot{q} = 0$ locus is

$$q = q^* + \left(\frac{\phi - \beta}{\phi}\right)(z - z^*).$$

Third, the $\dot{u} = 0$ locus is

$$u = u^* + \frac{q - q^*}{b}.$$

Based on these, it is easy to construct a phase diagram.[1] As signs of the slope of the $\dot{q} = 0$ locus depend on the comparison between ϕ and β, we must analyze the following three cases:

Case 1: $\phi > \beta$. According to Fig. A.3, $q(t)$ and $z(t)$ move in the same direction along the saddle path; put differently, they both increase or both decrease toward their respective steady-state values. At the same time, $q(t)$ and $u(t)$ move in the same direction along the transition path to the steady state. As such, whether $z(t), q(t), u(t)$ all start above or below their steady-state value, they monotonically converge to steady-state values z^*, q^*, u^*

Case 2: $\phi = \beta$. In this case, the $\dot{q} = 0$ locus is horizontal. So, as Fig. A.4 shows, $z(t)$ adjusts according to the globally stable equation $\dot{z}(t)/z(t) = -(1 - \beta)(z(t) - z^*)$ but with $q(t) = q^*$ and $u(t) = u^*$ at all times. Thus, the case of $\phi = \beta$ is like a freezing point; we have a *dynamic* system, but it is "frozen." For this reason, only when $\phi = \beta$, stochastic versions in the main text can be solved by hand.

Case 3: $\phi < \beta$. In this case, the $\dot{q} = 0$ locus slopes downward. But as Fig. A.5 shows, the dynamics is similar to that of A.3. The difference is that $q(t)$ and $z(t)$ move in opposite directions along the saddle path. So, $u(t)$ and $q(t)$ move in

[1] See Barro and Sala-i-Martin (2004, Ch.5) for a more comprehensive treatment. They prove that $z(t)$ adjusts monotonically from its initial value $z(0)$ to its steady state value z^*.

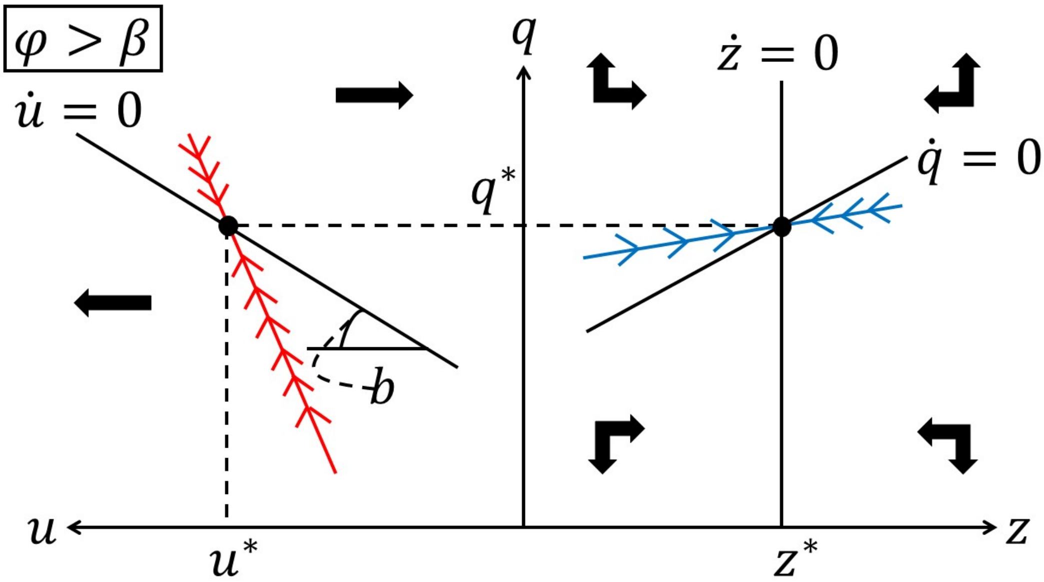

Figure A.3 Case 1: $\phi > \beta$.

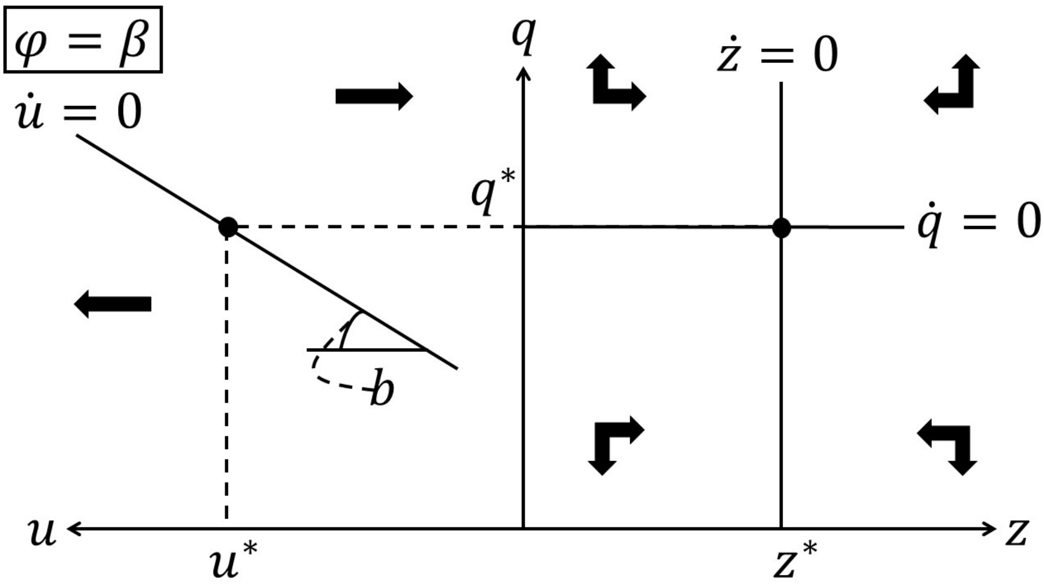

Figure A.4 Case 2: $\phi = \beta$.

the opposite direction of $z(t)$ along the transition path.

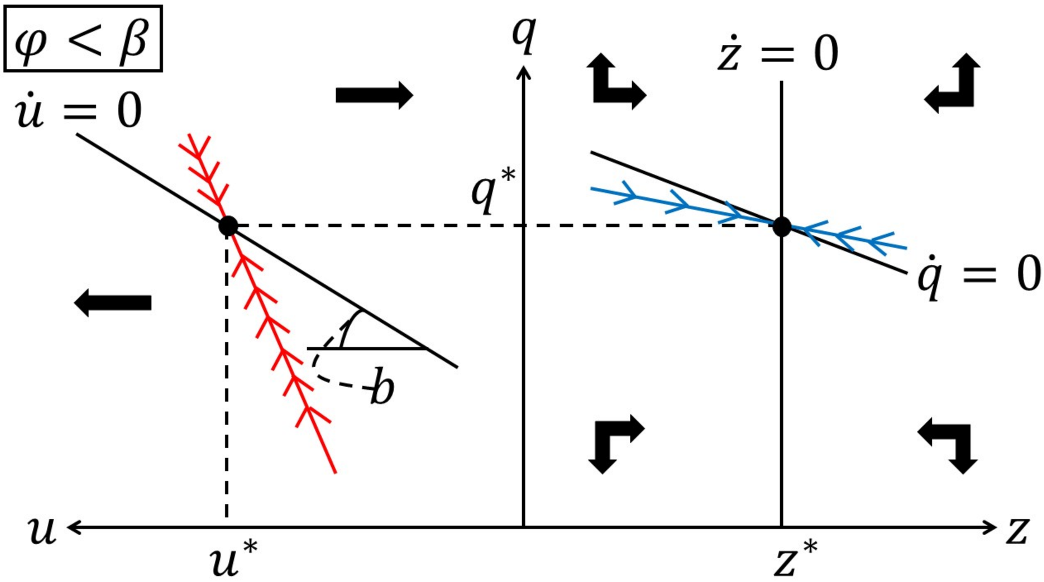

Figure A.5 Case 3: $\phi < \beta$.

To sum up, suppose $z(0) > z^*$ so that the initial return to the gross average product of physical capital higher than its steady-state counterpart. In this case, $z(0)$ gradually decreases toward z^*. In Case 1 ($\phi > \beta$), $u(t)$ and $q(t)$ monotonically decrease toward their steady-state values; in Case 2 ($\phi = \beta$), $u(t)$ and $q(t)$ remain constant respectively at u^* and q^*; and in Case 3 ($\phi < \beta$), $u(t)$ and $q(t)$ monotonically increase toward their steady-state values. Overall, consistent with the results of a Routh-Hurwitz stability criterion, we have established local (or saddle-path) stability.

A.7 Concluding Remarks

This Appendix presents a textbook Uzawa-Lucas model. It explains how to solve and understand the main properties of this model. Understanding a *deterministic* version will help you better understand stochastic versions in the main text; in particular, the reason why we need a Xie (1991; 1994) condition for our analytical inquiry into the growth-uncertainty nexus.

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