Families of canonical curves with genus 5 and the degenerations of the syzygies (II)

Takeshi Usa
Dept. of Math. Univ. of Hyogo *

Abstract

We consider a smooth affine curve B in the Hilbert scheme $Hilb_P^{A(m)}$ of $P = \mathbb{P}(\mathbb{C})^4$ associated with a Hilbert polynomial $A(m) = 8m - 4$, which includes all the canonical curves (i.e. non-singular projective and non-hyperelliptic curves of $g \geq 3$ embedded into projective spaces by their complete canonical linear systems) of genus 5 in its universal family. Assume that from the universal family of $Hilb_P^{A(m)}$, the curve B induces a family $f : \mathfrak{X} \to B$ of canonical curves with genus 5 and all the closed fibers are non-trigonal ones except only one trigonal closed fiber over a closed point $b_0 \in B$. In this article, we give a proof for an affirmative result on Conjecture 2.4 of [10], which claims that the structure of O_B-module $\mathscr{T}_3^{1,1}$ describing the first syzygies in degree 3 of the fibers can detect the transversality of the intersection at the point b_0 by the base curve B and a smooth branch of the divisor corresponding to the trigonal ones.

Keywords: canonical curve, genus 5, trigonal curve, degeneration of syzygies

§0 Introduction.

We slightly improved the results of [9] in the article [11] and found a technique to analyze the degeneration of q-th syzygies in any degree m by studying a coherent sheaf $\mathscr{T}_m^{1,q}$ on the base scheme, but at the lowest level on "q" (cf. Theorem 1.7 in [10], or [11]). In principle, this technique can be applied to any flat family of arithmetic D_2 closed subschemes with a limit fiber $X(b_0)$ which is a general projective scheme having the properties : $H^0(X(b_0), O_{X(b_0)}) \cong \mathbb{C}$ and $\dim X(b_0) > 0$. However, as our first case, we want to clarify essential difficulties in studying degeneration of syzygies. Thus we restricted ourselves to the case that the limit fiber $X(b_0)$ is a smooth projective variety with the degenerating syzygies, e.g. a trigonal canonical curve of genus 5. In the previous article [10], we gave a preparatory study on the degeneration of syzygies for a flat family of canonical curves of genus 5 over a smooth affine curve and presented Conjecture 2.4 in [10].

*2167 Shosha, Himeji, 671-2201 Japan.
E-mail address : usa@sci.u-hyogo.ac.jp
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For more precise descrption on this conjecture, let us consider the Hilbert scheme $\mathcal{H} = Hilb_P^{A(m)}$ of $P = \mathbb{P}(\mathbb{C})^4$ associated with a Hilbert polynomial $A(m) = 8m - 4$. The universal family $\mathcal{U} \to \mathcal{H}$ of $\mathcal{H}$ includes all the canonical curves of genus 5. Take a trigonal canonical curve $X \subseteq P$ and a closed point $b_0 = [X] \in Hilb_P^{A(m)}$ corresponding to the curve X. Then b_0 is a smooth point of $\mathcal{H}$. Set $\mathcal{D}$ to be a divisor which is a closure of the set of all the closed points in $\mathcal{H}$ corresponding to trigonal canonical curves with genus 5 in P. The divisor $\mathcal{D}$ has a (analytic local) smooth branch $\mathcal{D}_0$ at the point b_0. Then the normal direction $N_{\mathcal{D}_0/\mathcal{H},b_0}$ of the analytic local divisor $\mathcal{D}_0$ in $\mathcal{H}$ is described by $H^1(N_V \otimes I_{X/V})$, where V is a unique cubic surface including the curve X. We take a locally closed affine smooth curve B in $\mathcal{H}$ with the property $B \cap \mathcal{D} = \{b_0\}$ by removing other finite closed points of $B \cap \mathcal{D}$ from B if it is necessary. Then we obtain a flat and projective family $f : \mathfrak{X} = \mathcal{U} \times_{\mathcal{H}} B \to B$ of canonical curves with $g = 5$ over the curve B. For the projection morphism $\pi : P \times B \to B$, we consider the O_B-module structure of a higher direct image sheaf $\mathscr{T}_m^{p,q} = R^p\pi_*(\Omega^q_{P\times B/B} \otimes I_{\mathfrak{X}}(m))$ for the case $p = 1$, $q = 1$, and $m = 3$, which describes the degeneration of the first syzygies in degree $m = 3$. Then $Supp(\mathscr{T}_3^{1,1}) = \{b_0\}$ and $(\mathscr{T}_3^{1,1}) \otimes k(b_0) \cong k(b_0)^{\otimes 2}$. Since the tangent space $\Theta_{\mathcal{H},b_0}$ of $\mathcal{H}$ at the point b_0 is described by $H^0(N_X)$, the curve B determines a normal vector field $\sigma \in H^0(N_X)$ as its tangent vector in $\Theta_{\mathcal{H},b_0}$.

The natural map $\Theta_{\mathcal{H},b_0} \to N_{\mathcal{D}_0/\mathcal{H},b_0}$ corresponds to the composition map $\tau : H^0(N_X) \to H^0(N_V \otimes O_X) \to H^1(N_V \otimes I_{X/V})$. Conjecture 2.4 in [10] insists that if $\tau(\sigma) \neq 0$, then the sheaf $\mathscr{T}_3^{1,1}$ itself is isomorphic to $k(b_0)^{\oplus 2}$.

In this article, we give a proof of this conjecture via infinitesimal study of embedded deformation of the curve X in P. Thus, this work might be considered as a partial review of classical works [4], [5], and [6] from the view point of infinitesimal study on the Hilbert schemes.

We refer fundamentally to [10], [2] or [1], and often use the terminology and the results in [10], or in [2] without mentioning except somethings important.

§1 Main results.

On Conjecture 2.4 in [10], we have an affirmative result as follows.

Main Theorem 1.1 *Since $Supp(\mathscr{T}_3^{1,1}) = \{b_0\}$ and $\mathscr{T}_3^{1,1} \otimes k(b_0) \cong k(b_0)^{\oplus 2}$, we set $\mathscr{T}_3^{1,1} \cong O_{B,b_0}/(t^{k_1}) \oplus O_{B,b_0}/(t^{k_2})$ by using a regular parameter "t" of O_{B,b_0} and a map τ to be a composition map $H^0(N_X) \to H^0(N_V \otimes O_X) \to H^1(N_V \otimes I_{X/V})$, where V denotes a unique non-singular cubic surface in $P = \mathbb{P}^4(\mathbb{C})$ which includes the trigonal curve X. Then, $k_1 = k_2 = 1$ if and only if $\tau(\sigma) \neq 0 \in H^1(N_V \otimes I_{X/V}) \cong N_{\mathcal{D}_0/\mathcal{H},b_0}$.*

Proof. In Main Theorem 1.1 above, it is rather easy to show the "only if" part that $\tau(\sigma) = 0$ implies that $k_1 \geq 2$ and $k_2 \geq 2$. The condition $\tau(\sigma) = 0$ implies that the section σ can be lifted to a section $\widetilde{\sigma} \in H^0(N_{(X,V)})$, which determines a tangent vector $v_{\widetilde{\sigma}} \in \Theta_{\mathscr{F},([X],[V])}$ of the flag Hilbert scheme $\mathscr{F}$ at the closed point $([X],[V])$. Since the flag Hilbert scheme $\mathscr{F}$ is a projective scheme and is smooth around the closed point $([X],[V])$, we can easily find a smooth affine curve C passing through the point $([X],[V])$ in the direction $v_{\widetilde{\sigma}}$. By the universality of the flag Hilbert scheme, we have a flag-family $f' : \mathfrak{X}' \to C$ and $g' : \mathfrak{V}' \to C$ which are compatible with the inclusion $\mathfrak{X}' \subseteq \mathfrak{V}'$. Replacing the curve C by a sufficiently small open set, we may assume that all the fibers of f' are smooth and all the fibers of g' are irreducible surfaces of degree 3. Hence all the fibers of f' are trigonal canonical curves of genus 5, and therefore the family $f' : \mathfrak{X}' \to C$ is a Betti constant family. Thus, by restricting this flag family $f' : \mathfrak{X}' \to C$ and $g' : \mathfrak{V}' \to C$ to the first infinitesimal neighborhood $\overline{C}_1$ of the closed point $([X],[V])$, we obtain an infinitesimal flag family $\overline{f'} : \overline{\mathfrak{X}'} \to \overline{C}_1$ and $\overline{g'} : \overline{\mathfrak{V}'} \to \overline{C}_1$ which are compatible with the inclusion $\overline{\mathfrak{X}'} \subseteq \overline{\mathfrak{V}'}$, which is isomorphic to the flag family induced from the section $\widetilde{\sigma} \in H^0(N_{(X,V)})$. Since the section $\widetilde{\sigma}$ is a lift of the section $\sigma \in H^0(N_X)$, the infinitesimal families $\overline{f'} : \overline{\mathfrak{X}'} \to \overline{C}_1$ and $\overline{f} : \overline{\mathfrak{X}} \to \overline{B}_1$ are isomorphic with each other. Then, the Betti constancy of the family $f' : \mathfrak{X}' \to C$ implies that of the infinitesimal family $\overline{f} : \overline{\mathfrak{X}} \to \overline{B}_1$. This shows that the module $\mathscr{T}_3^{1,1} \otimes O_{\overline{B}_1} \cong \overline{\mathscr{T}}_{3,1}^{1,1}$ is an $O_{\overline{B}_1}$-(locally) free module $O_{\overline{B}_1} \oplus O_{\overline{B}_1}$, namely $\mathscr{T}_3^{1,1} \cong O_{B,b_0}/(t^{k_1}) \oplus O_{B,b_0}/(t^{k_2})$ with $k_1,\ k_2 \geq 2$.

On the other hand, the converse direction, namely to show that $\tau(\sigma) \neq 0$ implies $k_1 = k_2 = 1$ is a rather troublesome part in our proof. Now we recall our results in [8]. From the first row in the diagram (#-3) of $O_{P\times\overline{B}_1}$-modules in [8], namely the sequence :

$$0 \longrightarrow I_X \xrightarrow{\times\varepsilon} I_{\overline{\mathfrak{X}}} \longrightarrow I_X \longrightarrow 0 \qquad \text{(\#-1)}$$

with tensoring the $O_{P\times\overline{B}_1}$-locally free module $\Omega^1_{P\times\overline{B}_1/\overline{B}_1}(3)$, we get a long exact sequence :

$$0 \to T_3^{0,1}(b_0) \to (\overline{\mathscr{T}}_{3,1}^{0,1})_{b_0} \xrightarrow{\lambda} T_3^{0,1}(b_0) \xrightarrow{ob_{\widetilde{\sigma}}} T_3^{1,1}(b_0) \xrightarrow{\mu} (\overline{\mathscr{T}}_{3,1}^{1,1})_{b_0} \to T_3^{1,1}(b_0) \to 0. \qquad \text{(\#-2)}$$

Following the principle of Theorem 1.4 in [10], it is enough to show the surjectivity of the obstruction map ob_σ in the sequence (#-2). From the sequence (#-1) and a canonical injective homomorphism $I_V \to I_X$ of sheaves, we take a fiber product sheaf $I_{\overline{\mathfrak{X}}} \times_{I_X} I_V$, which induces a natural exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & I_{X/V} & = & I_{X/V} & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_X & \xrightarrow[\alpha_1]{\times\varepsilon} & I_{\overline{\mathfrak{X}}} & \xrightarrow[\beta_1]{} & I_X & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s''=incl.} & & \\
0 & \longrightarrow & I_X & \xrightarrow[\alpha_2]{} & I_{\overline{\mathfrak{X}}} \times_{I_X} I_V & \xrightarrow[\beta_2]{} & I_V & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0. & &
\end{array} \qquad \text{(\#-3)}$$

Tensoring an $O_{P\times\overline{B}_1}$-locally free sheaf $\Omega^1_{P\times\overline{B}_1/\overline{B}_1}(3)$ to the diagram (#-3) above, we have:

$$
\begin{array}{ccccccc}
0 \longrightarrow & H^0(I_V \otimes \Omega^1_P(3)) & \xrightarrow{\cong} & H^0(I_X \otimes \Omega^1_P(3)) = T_3^{0,1}(b_0) & \longrightarrow & H^0(I_{X/V} \otimes \Omega^1_P(3)) = 0 \\
& \delta_{II}\downarrow & & \downarrow \delta_{I}:=ob_\sigma & & \\
& H^1(I_X \otimes \Omega^1_P(3)) & = & H^1(I_X \otimes \Omega^1_P(3)) = T_3^{1,1}(b_0), & &
\end{array}
\qquad (\#\text{-}4)
$$

which implies $Coker(ob_\sigma) \cong Coker(\delta_{II})$. Here, to see $H^0(I_{X/V} \otimes \Omega^1_P(3)) = 0$, we only need to recall the facts that $H^0(I_{X/V} \otimes \Omega^1_P(3)) \subseteq \oplus H^0(I_{X/V}(2))$, the surface V is arithmetically Cohen-Macaulay, hence $H^1(I_V(*)) = 0$, and the surface V is a quadric hull of X, namely $H^0(I_V(2)) \cong H^0(I_X(2))$, and therefore $H^0(I_{X/V}(2)) = 0$.

As the next step, starting from the sequence $0 \to I_X \to I_{\overline{\mathfrak{X}}} \times_{I_X} I_V \to I_V \to 0$ in the diagram (#-3), we set the $O_{P\times \overline{B}_1}$ -submodule M to be $Coker[I_V \to I_{\overline{\mathfrak{X}}} \times_{I_X} I_V]$ via $I_V \subset I_X \subset I_{\overline{\mathfrak{X}}} \times_{I_X} I_V$ and obtain an exact commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & I_{X/V} & \xrightarrow{\alpha_3} & M & \xrightarrow{\beta_3} & I_V & \longrightarrow & 0 \\
 & & r'\uparrow & & \uparrow r & & \| & & \\
0 & \longrightarrow & I_X & \xrightarrow[\alpha_2]{} & I_{\overline{\mathfrak{X}}} \times_{I_X} I_V & \xrightarrow[\beta_2]{} & I_V & \longrightarrow & 0. \\
 & & incl.=u'\uparrow & & \uparrow u & & & & \\
 & & I_V & = & I_V & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\qquad (\#\text{-}5)
$$

Now we have to make a remark that all the modules in (#-5) except the module $I_{\overline{\mathfrak{X}}} \times_{I_X} I_V$ are annihilated by $\times\varepsilon$, and therefore they are O_P-modules via $O_P \cong O_{P\times \overline{B}_1}/\varepsilon O_{P\times \overline{B}_1}$. Again tensoring the $O_{P\times \overline{B}_1}$-locally free sheaf $\Omega^1_{P\times \overline{B}_1/\overline{B}_1}(3)$ to the diagram (#-5), we get :

$$
\begin{array}{ccccccc}
 & & H^0(I_V \otimes \Omega^1_P(3)) & = & H^0(I_V \otimes \Omega^1_P(3)) & & \\
 & & \delta_{II}\downarrow & & \downarrow \delta_{III} & & \\
0 = H^1(I_V \otimes \Omega^1_P(3)) & \longrightarrow & H^1(I_X \otimes \Omega^1_P(3)) & \xrightarrow[\cong]{\beta_{X/V}} & H^1(I_{X/V} \otimes \Omega^1_P(3)) & \longrightarrow & 0
\end{array}
\qquad (\#\text{-}6)
$$

This shows that $Coker(\delta_{II}) \cong Coker(\delta_{III})$. Here, the surjectivity of the map $\beta_{X/V}$ is ensured by the fact that the surface V is a homological shell of X. To see $H^1(I_V \otimes \Omega^1_P(3)) = 0$, we have only to remind that the surface V is arithmetically Cohen-Macaulay and its ideal is generated only by quadric equations.

From the construction of the module M, we can see that the module M is a submodule of $I_V \cdot 1 \oplus O_V \cdot \varepsilon$ and has a stalk-wise expression:

$$M=\{(a,c\varepsilon)\in\ I_V\cdot 1\oplus O_V\cdot\varepsilon\ \ |\ \ a\in I_V,\ c\in O_V,\ \text{s.}t.\ \ \bar{c}=-\sigma|_V(a)\ \ \text{in } O_X\}. \tag{\#-7}$$

Then, a homomorphism of sheaves $g: I_V^2\ni x\mapsto(x\cdot 1,\ 0\cdot\varepsilon)\in I_V\cdot 1\oplus O_V\cdot\varepsilon$ in a stalk-wise expression has its image in M, which induces an exact commutative diagram of O_P-modules:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_{X/V} & \xrightarrow{\alpha_4} & \overline{M} & \xrightarrow{\beta_4} & I_V/I_V^2 & \longrightarrow & 0 \\
 & & \| & & h\uparrow & & \uparrow h'' & & \\
0 & \longrightarrow & I_{X/V} & \underset{\alpha_3}{\longrightarrow} & M & \underset{\beta_3}{\longrightarrow} & I_V & \longrightarrow & 0. \\
 & & & & g\uparrow & & \uparrow g'' & & \\
 & & & & I_V^2 & = & I_V^2 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{\#-8}$$

By tensoring $\Omega_P^1(3)$ to the diagram (#-8) above, we have

$$\begin{array}{ccccccc}
0=H^0(\Omega_P^1(3)\otimes I_V^2) & \longrightarrow & H^0(\Omega_P^1(3)\otimes I_V) & \xrightarrow{\cong} & H^0(\Omega_P^1(3)\otimes I_V/I_V^2) & \longrightarrow & H^1(\Omega_P^1(3)\otimes I_V^2)=0 \\
 & & \delta_{III}\downarrow & & \downarrow\delta_{IV} & & \\
 & & H^1(\Omega_P^1(3)\otimes I_{X/V}) & = & H^1(\Omega_P^1(3)\otimes I_{X/V}) & & \\
 & & & & \downarrow & & \\
 & & & & H^1(\Omega_P^1(3)\otimes\overline{M}). & &
\end{array} \tag{\#-9}$$

By Claim 1.5 and Claim 1.8 below, we see that $Coker(\delta_{III})\cong Coker(\delta_{IV})$ and $Coker(\delta_{IV})\subseteq H^1(\Omega_P^1(3)\otimes\overline{M})$. Then, Claim 1.9 shows that $H^1(\Omega_P^1(3)\otimes\overline{M})=0$, which implies $Coker(ob_\sigma)\cong Coker(\delta_{IV})=0$, namely the surjectivity of the map ob_σ.

Claim 1.2 *For the conormal bundle I_V/I_V^2 of the surface V, we have a short exact sequence:*

$$0\longleftarrow I_V/I_V^2\longleftarrow O_V(-2)^{\oplus 3}\longleftarrow O_V(-5\xi-3\varepsilon)\longleftarrow 0. \tag{\#-10}$$

Proof. Since the surface V is a Hirzebruch surface F_1, namely a one point blow-up of $\mathbb{P}^2$, which is arithmetically Cohen-Macaulay and a variety of minimal degree in $\mathbb{P}^4=Proj(S)$, whose homogeneous ideal $\mathbb{I}_V$ is generated by three quadric equations $\{G_1,G_2,G_3\}$, we have a short exact sequence of the sheaves :

$$0\longleftarrow I_V\longleftarrow O_P(-2)^{\oplus 3}\longleftarrow O_P(-3)^{\oplus 2}\longleftarrow 0 \tag{\#-11}$$

from a minimal graded S-free resolution of $\mathbb{I}_V$. Tensoring O_V to the sequence (#-11) and putting $L := Im[O_V(-3)^{\oplus 2} \to O_V(-2)^{\oplus 3}]$, we see that the sheaf L is a line bundle on V and $c_1(L) = -5\xi - 3\varepsilon$ by the reason that $O_V(1) \cong O_V(2\xi + \varepsilon)$ and $det(I_V/I_V^2) = O_V(-7\xi - 3\varepsilon)$. ∎

Claim 1.3

$$H^0(I_V(2)) \xrightarrow{\cong} H^0((I_V/I_V^2)(2)), \qquad (\#\text{-}12)$$

Proof. Compare the two exact sequences (#-10) and (#-11) after tensoring $O_P(2)$.

$$\begin{array}{ccccccccc} 0 & \longleftarrow & I_V(2) & \longleftarrow & O_P^{\oplus 3} & \longleftarrow & O_P(-1)^{\oplus 2} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & (I_V/I_V^2)(2) & \longleftarrow & O_V^{\oplus 3} & \longleftarrow & O_V(-\xi-\varepsilon) & \longleftarrow & 0. \end{array} \qquad (\#\text{-}13)$$

Take the cohomologies of the sheaves in (#-13) and see the result (#-12) from the exact commutative diagram:

$$\begin{array}{ccccccc} 0 = H^1(O_P(-1))^{\oplus 2} & \longleftarrow & H^0(I_V(2)) & \xleftarrow{\cong} & H^0(O_P)^{\oplus 3} & \longleftarrow & H^0(O_P(-1))^{\oplus 2} = 0 \\ & & \downarrow & & \downarrow \cong & & \\ 0 = H^1(O_V(-\xi-\varepsilon)) & \longleftarrow & H^0((I_V/I_V^2)(2)) & \underset{\cong}{\longleftarrow} & H^0(O_V)^{\oplus 3} & \longleftarrow & H^0(O_V(-\xi-\varepsilon)) = 0. \end{array}$$

(#-14)

∎

Claim 1.4

$$H^0(I_V^2(2)) = H^1(I_V^2(2)) = 0 \qquad (\#\text{-}15)$$

Proof. Consider the exact commutative diagram of sheaves :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_V(2) & \longrightarrow & O_P(2) & \longrightarrow & O_V(2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (I_V/I_V^2)(2) & \longrightarrow & (O_P/I_V^2)(2) & \longrightarrow & O_V(2) & \longrightarrow & 0. \end{array} \qquad (\#\text{-}16)$$

Taking their cohomologies with using Claim 1.3 :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(I_V(2)) & \longrightarrow & H^0(O_P(2)) & \longrightarrow & H^0(O_V(2)) & \longrightarrow & 0, \\ & & \cong\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^0((I_V/I_V^2)(2)) & \longrightarrow & H^0(O_P/I_V^2)(2)) & \longrightarrow & H^0(O_V(2)) & & \end{array} \qquad (\#\text{-}17)$$

we obtain $H^0(O_P(2)) \cong H^0(O_P/I_V^2)(2))$. Then the long exact sequence :

$$0 \longrightarrow H^0(I_V^2(2)) \longrightarrow H^0(O_P(2)) \xrightarrow{\cong} H^0((O_P/I_V^2)(2))$$

$$\longrightarrow H^1(I_V^2(2)) \longrightarrow H^1(O_P(2)) = 0$$

brings the result (#-15). ∎

Claim 1.5

$$H^0(\Omega_P^1(3) \otimes I_V^2) = 0 \tag{\#-18}$$

Proof. Now it is easy to see Claim 1.5 since $H^0(\Omega_P^1(3) \otimes I_V^2) \subseteq \oplus H^0(I_V^2(2))$ by the Euler sequence and Claim 1.4.

∎

We recall the notation in [7] and use them in our proves of the following several claims.

Claim 1.6 *For an ideal sheaf J of O_P and a non-negative integer m, in the following exact commutative diagram* (#-19)*, we have*

$$-m \cdot \delta_{EN} = \delta_{LFT} \circ \overline{d_J} \circ canl. \quad ,$$

where the map "canl." denotes the canonical homomorphism.

$$\begin{array}{ccccc} H^0(J(m)) & \xrightarrow{-m\cdot\delta_{EN}} & H^1(\Omega_P^1(m) \otimes J) & \longrightarrow & \oplus H^1(J(m-1)) \\ canl.\downarrow & & \uparrow \delta_{LFT} & & \\ H^0(J/J^{3/2}(m)) & \xrightarrow[\overline{d_J}]{} & H^0(\Omega_P^1(m) \otimes O_P/J) & & \\ & & \uparrow \beta_{LFT} & & \\ & & H^0(\Omega_P^1(m)) & & \end{array} \tag{\#-19}$$

Here $J^{3/2}$ denotes the kernel sheaf of the natural O_P-linear sheaf homomorphism $d_J : J \to \Omega_P^1 \otimes O_P/J$.

Proof. See (2.2) Lemma in [7]. ∎

Claim 1.7

$$H^1(I_V/I_V^2(\xi)) = 0. \tag{\#-20}$$

Proof. Recall the sequence (#-10) with tensoring $O_V(\xi)$:

$$0 \longleftarrow I_V/I_V^2(\xi) \longleftarrow \oplus^3 O_V(-3\xi - 2\varepsilon) \longleftarrow O_V(-4\xi - 3\varepsilon) \longleftarrow 0, \tag{\#-21}$$

which induces a cohomology exact sequence :

$$H^1(I_V/I_V^2(\xi)) \longleftarrow \oplus^3 H^1(O_V(-3\xi-2\varepsilon)) = 0$$

$$\oplus^3 H^2(O_V(-3\xi-2\varepsilon)) \underset{\mu}{\longleftarrow} H^2(O_V(-4\xi-3\varepsilon)) \longleftarrow \qquad (\#\text{-}22)$$

where $H^1(O_V(-3\xi-2\varepsilon)) = 0$ is shown by the Serre duality and $H^1(O_V) = 0$. Thus, it is enough to see the injectivity of the map $\mu : H^2(O_V(-4\xi-3\varepsilon)) \to \oplus^3 H^2(O_V(-3\xi-2\varepsilon))$. It is equivalent to show the surjectivity of the map $\mu^\vee \oplus^3 H^0(O_V) \to H^0(O_V(\xi+\varepsilon))$ by the Serre duality. Take the dual of the sequence (#-21) with tensoring $O_V(K_V)$, we have

$$0 \longrightarrow N_V(-4\xi-2\varepsilon) \longrightarrow \oplus^3 O_V \underset{\beta_V}{\longrightarrow} O_V(\xi+\varepsilon) \longrightarrow 0. \qquad (\#\text{-}23)$$

The map $\mu^\vee$ arises from the sheaf homomorphism β_V which is given by three sections $\ell_1,\ \ell_2,\ \ell_3 \in H^0(O_V(\xi+\varepsilon))$.

Since the surface V is a one point blow up of the projective plane $Y = \mathbb{P}^2$, the three sections $\ell_1,\ \ell_2,\ \ell_3$ comes from three lines in $\mathbb{P}^2$ via $H^0(O_V(\xi+\varepsilon)) \cong H^0(O_Y(1))$. The three sections $\ell_1,\ \ell_2,\ \ell_3$ have to generate the line bundle $O_V(\xi+\varepsilon)$, or have no base point. Thus three sections $\ell_1,\ \ell_2,\ \ell_3$ are linearly independent sections in $H^0(O_V(\xi+\varepsilon))$, which implies the surjectivity of the map $\mu^\vee$. ∎

Claim 1.8

$$H^1(\Omega_P^1(3) \otimes I_V^2) = 0 \qquad (\#\text{-}24)$$

Proof. We apply Claim 1.6 by putting $J = I_V^2$ and $m = 3$. Then $J^{3/2} = I_V^3$ and Claim 1.4 show that

$$\begin{array}{ccccccc}
0 = \oplus H^0(I_V^2(2)) & \xrightarrow{\beta_{EN}} & H^0(I_V^2(3)) & \underset{\cong}{\xrightarrow{-3\cdot\delta_{EN}}} & H^1(\Omega_P^1(3)\otimes I_V^2) & \xrightarrow{\alpha_{EN}} & \oplus H^1(I_V^2(2)) = 0 \\
 & & {\scriptstyle canl.}\downarrow & & \uparrow{\scriptstyle \delta_{LFT}} & & \\
 & & H^0(I_V^2/I_V^3(3)) & \underset{d_{I_V^2}}{\longrightarrow} & H^0(\Omega_P^1(3)\otimes O_P/I_V^2). & &
\end{array} \qquad (\#\text{-}25)$$

If we see that $H^0(I_V^2/I_V^3(3)) = 0$, then Claim 1.6 implies that the map $-3\cdot\delta_{EN}$ is a zero map, namely the image $Im(-3\cdot\delta_{EN}) = H^1(\Omega_P^1(3)\otimes I_V^2)$ is zero, which was we want to show in Claim 1.8.

Let us show $H^0(I_V^2/I_V^3(3)) = 0$ in the sequel. Since $I_V^2/I_V^3(3) \cong Sym^2(I_V/I_V^2)(3)$ and $Sym^2(I_V/I_V^2)(3)$ is a direct summand of the sheaf $I_V/I_V^2 \otimes I_V/I_V^2(3)$, it is enough to show $H^0(I_V/I_V^2 \otimes I_V/I_V^2(3)) = 0$. Recall the sequence (#-10) with tensoring $I_V/I_V^2(3) \cong I_V/I_V^2(6\xi+3\varepsilon)$:

$$0 \longleftarrow I_V/I_V^2 \otimes I_V/I_V^2(3) \longleftarrow \oplus^3 I_V/I_V^2(1) \longleftarrow I_V/I_V^2(\xi) \longleftarrow 0, \qquad (\#\text{-}26)$$

which implies an exact sequence : $H^1(I_V/I_V^2(\xi)) \leftarrow H^0(I_V/I_V^2 \otimes I_V/I_V^2(3)) \leftarrow \oplus^3 H^0(I_V/I_V^2(1)) = 0$. Then Claim 1.7 shows $H^0(I_V/I_V^2 \otimes I_V/I_V^2(3)) = 0$. ∎

Claim 1.9

$$\overline{M} \cong O_V(-2)^{\oplus 3}, \quad Coker(\delta_{IV}) \subseteq H^1(\Omega^1_P(3) \otimes \overline{M}) = 0.$$

Proof. Let us recall a short exact sequence:

$$0 \longrightarrow I_{X/V} \xrightarrow{\alpha_4} \overline{M} \xrightarrow{\beta_4} I_V/I_V^2 \longrightarrow 0 \tag{\#-27}$$

in the exact commutative diagram (#-8). To show that the sequence (#-27) does not split, we assume that there exists an O_P-linear homomorphism $\overline{\rho} : \overline{M} \to I_{X/V}$ which gives a splitting of the sequence (#-27), namely $\overline{\rho} \circ \alpha_4 = 1_{I_{X/V}}$. Then we set an O_P-linear homomorphism $\rho : M \to I_{X/V}$ to be $\rho := \overline{\rho} \circ h$ in the diagram (#-8). Then, $\rho \circ \alpha_3 = \overline{\rho} \circ h \circ \alpha_3 = \overline{\rho} \circ \alpha_4 = 1_{I_{X/V}}$. Thus we have a splitting of the sequence:

$$0 \longrightarrow I_{X/V} \xrightarrow{\alpha_3} M \xrightarrow{\beta_3} I_V \longrightarrow 0 \tag{\#-28}$$

by the O_P-linear homomorphism $\rho : M \to I_{X/V}$. Put an O_P-submodule K of M to be $K = Ker(\rho) \subseteq M$. Obviously the O_P-module K is isomorphic to the O_P-module I_V via an O_P-linear homomorphism β_K which is a restriction of β_3 to the O_P-submodule K of M. Now we consider the module K to be an $O_{P\times\overline{B}_1}$-module which is annihilated by ε. Since the homomorphism "r" in the diagram (#-5) is $O_{P\times\overline{B}_1}$-linear, we obtain an $O_{P\times\overline{B}_1}$-submodule $\widetilde{K}$ of $I_{\overline{\mathfrak{X}}} \times_{I_X} I_V$ by $\widetilde{K} := r^{-1}(K) \cong (I_{\overline{\mathfrak{X}}} \times_{I_X} I_V) \times_M K$, which induces a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_X & \xrightarrow[\alpha_1]{\times\varepsilon} & I_{\overline{\mathfrak{X}}} & \xrightarrow[\beta_1]{} & I_X & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s''=incl.} & & \\
0 & \longrightarrow & I_X & \xrightarrow[\alpha_2]{} & I_{\overline{\mathfrak{X}}} \times_{I_X} I_V & \xrightarrow[\beta_2]{} & I_V & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle incl.=u'} & & \| & & \uparrow{\scriptstyle \beta_3} & & \\
0 & \longrightarrow & I_V & \xrightarrow[u]{} & I_{\overline{\mathfrak{X}}} \times_{I_X} I_V & \xrightarrow[r]{} & M & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \widetilde{\iota}=incl.} & & \uparrow{\scriptstyle \iota=incl.} & & \\
0 & \longrightarrow & I_V & \xrightarrow[u]{} & \widetilde{K} & \xrightarrow[r]{} & K & \longrightarrow & 0,
\end{array} \tag{\#-29}$$

where all the horizontal lines are exact. Since the homomorphism $\beta_K = \beta_3 \circ \iota$ is the O_P-linear isomorphism, we obtain an $O_{P\times\overline{B}_1}$-linear exact commutative diagram :

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_{X/V} & \xrightarrow[\alpha_5]{\times\varepsilon} & I_{\overline{\mathfrak{X}}}/\widetilde{K} & \xrightarrow[\beta_5]{} & I_{X/V} & \longrightarrow & 0 \\
 & & {\scriptstyle r'}\uparrow & & \uparrow & & \uparrow{\scriptstyle r'} & & \\
0 & \longrightarrow & I_X & \xrightarrow[\alpha_1]{\times\varepsilon} & I_{\overline{\mathfrak{X}}} & \xrightarrow[\beta_1]{} & I_X & \longrightarrow & 0 \\
 & & {\scriptstyle incl.=u'}\uparrow & & {\scriptstyle s\circ\widetilde{\iota}}\uparrow & & \uparrow{\scriptstyle s''=incl.} & & \\
0 & \longrightarrow & I_V & \xrightarrow[u]{} & \widetilde{K} & \xrightarrow[\beta_3\circ\iota\circ r]{} & I_V & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad (\#\text{-}30)
$$

Put $I_{\overline{\mathfrak{Y}}} := \widetilde{K} \subseteq I_{\overline{\mathfrak{X}}} \subseteq O_{P\times\overline{B}_1}$ to be an ideal sheaf of a closed subscheme $\overline{\mathfrak{Y}} \subseteq P \times \overline{B}_1$, which is flat over $\overline{B}_1$ by using the flatness of $O_{\overline{\mathfrak{X}}}$ and of $I_{\overline{\mathfrak{X}}/\overline{\mathfrak{Y}}} = I_{\overline{\mathfrak{X}}}/\widetilde{K}$ over $\overline{B}_1$ (cf. [3] Proposition 2.2) which is guaranteed by the fact that the natural stalk-wise homomorphism $\alpha_{5,p} \otimes k(b_0) : I_{X/V,p} \otimes k(b_0) \to (I_{\overline{\mathfrak{X}}}/\widetilde{K})_p \otimes k(b_0)$ at each point $p \in P$ is zero since $\alpha_{1,p} \otimes k(b_0)$ is zero. Then the pair $\overline{\mathfrak{X}} \subseteq \overline{\mathfrak{Y}}$ gives an 1-st infinitesimal embedded deformation of the pair $X \subseteq V$ in the space P, which implies that the section $\sigma \in H^0(N_X)$ has a lifting $\widetilde{\sigma} \in H^0(N_{(X,V)})$, namely $\tau(\sigma) = 0 \in H^1(N_V \otimes I_{X/V})$, which is a contradiction. Thus we see that the module $\overline{M}$ in the sequence (#-27) gives a non-trivial O_P-module extension of I_V/I_V^2 by $I_{X/V}$.

Now we recall the sequence (#-10), which is obviously a non-trivial O_P-module extension of I_V/I_V^2 by $O_V(-5\xi - 3\varepsilon) \cong I_{X/V}$. Since $\dim_{\mathbb{C}} Ext^1_{O_V}(I_V/I_V^2, I_{X/V}) = h^1(N_V \otimes I_{X/V}) = 1$ (cf. Theorem 2.1 in [10]), each of the two extension classes of the sequences (#-10) and (#-27) gives a base of the 1-dimensional vector space, which implies the equivalence of the both extensions and $\overline{M} \cong O_V(-2)^{\oplus 3}$. Then $H^1(\Omega^1_P(3) \otimes \overline{M}) \cong \oplus^3 H^1(\Omega^1_P \otimes O_V(1))$, which is zero by using the linear normality coming from the arithmetically Cohen-Macaulay property of the surface V. ∎

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