Capital-Skill Complementarity, Biased Technological Change, and Balanced Growth

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Abstract

Grossman et al. (2017a) shows that balanced growth path (BGP) exists even when the elasticity of substitution between factors is not unity and capital-biased technological change occurs. However, their model is ambiguous in the expression of education. Our studies show that three-factor production function of capital, skilled labor and unskilled labor which is specified as a Two-Level CES function can provide some conclusion. It can also represent the extent supported by empirical studies of capital-skill complementarity for elasticity of substitution between factors.

Keywords: Skilled Labor, Capital-Skill Complementarity, Biased Technological Change, Balanced Growth
(JEL : E22, E24, E25, J22, J24, O33, O41)

1. Introduction

Kaldor (1961) stated that output per worker and capital per worker have grown steadily, and, while the capital-output ratio, the real return on capital, and the shares of capital and labor in national income[1] have remained fairly constant. In recent study, Jones (2016) reports that real per capita GDP in the United States has grown at remarkably steady average rate of around two percent per year for a period of nearly 150 years, while the ratio of capital to output has remained nearly constant. These facts suggest to many the relevance of a balanced growth path (BGP) which experiences constant proportional rates of growth of output and consumption. There are studies to establish models that predict them.

Uzawa (1961) pointed out that balanced growth in a neoclassical economy with

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[1] The share of labor has declined in recent years. See Atkinson, Piketty and Saez (2011) ,Elsby, Hobijn and ŞAhİN (2013), and Piketty and Zucman (2014).

exogenous population growth requires either production function with a unitary elasticity of substitution between capital and labor σ_{KL} or an absence of capital-augmenting technological progress. This was proved by Schlicht (2006). Uzawa (1965) and Lucas (1988) showed existence of BGP with human capital investment, and their model have not capital-augmenting technological progress.

There are many empirical studies[2] on $\sigma_{KL} \neq 1$. Also, price of investment goods are decreasing[3]. This implies capital-augmenting technological progress. From above, there are many empirical studies that the conditions for establishing BGP are not satisfied.

Based on the problem awareness mentioned above, Grossman et al. (2017a) extend Uzawa's theorem by embedded education index. They shows that balanced growth path (BGP) exists even when the elasticity of substitution between factors is not unity and capital-biased technological change occurs if capital is more complementary with schooling than with raw labor.

Grossman et al. (2017a) define schooling as scholar variable[4], and they consider two factors production function of capital and labor. The increase in education level in their model merely represents an increase in productivity, and the reward for education is unclear. It also shows capital-complementarity when the elasticity of substitution between capital and labor is less than one. However, the empirical study of capital-skill complementarity[5] supports that the elasticity of substitution between capital and skilled labor is less than 1, while the elasticity of substitution between capital and unskilled labor is greater than 1. Therefore, in their model, the wage gap is ambiguous..

In the study of capital-skill complementarity, there are many discussions using a production function with three or more factors, including a Two-Level Constant of Elasticity Substitution (Two-Level CES) production function. Bowles (1970) estimated the elasticity of substitution among labor. Goldin and Katz (1998) showed capital-skill complementarity in the prewar U.S. manufacturing. Krusell et al. (2000) estimated the

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[2] See Chirinko (2008) Chirinko, Fazzari and Meyer (2011), and Chirinko and Mallick (2017) for empirical studies showing $\sigma_{KL} < 1$. See Karabarbounis and Neiman (2014) for an empirical study showing $\sigma_{KL} > 1$.

[3] See Greenwood, Hercowitz and Krusell (1997) and Jones (2016).Grossman et al. (2017a)showed decline of price of investment goods by Federal Reserve Economic Data (FRED).

[4] Grossman et al. (2017b) and Grossman et al. (2020) also incorporated education into Uzawa's growth theorem to show the existence of BGP and then discuss the factor share. However, their formulation is also ambiguous in terms of education and capital-skill complementarity.

[5] See Goldin and Katz (1998), Krusell et al. (2000) and Papageorgiou and Saam (2008) for empirical studies on capital-skill-complementarity. Also, see Hidalgo Pérez, O'Kean Alonso and Rodríguez López (2016) and McAdam and Willman (2018) for recent empirical studies on capital-skill complementarity.

elasticity of substitution between factors from U.S. data 1963-1992, and showed capital-skill complementarity. Papageorgiou and Saam (2008) theoretically analyze the existence and stability of steady state under capital-skill complementarity. In above studies, they used Two-Level CES production function, and production factors are three or more. Therefore, in order to incorporate the education level of workers into the model and represent the capital-skill complementarity, we should discuss with a production function with three or more factors.

The purpose of this paper is to show that BGP exists even when the elasticity of substitution between production factors is not 1 and capital-augmenting technological progress occurs under the setting of the three-factor production function. In this article, following Grossman et al. (2017a), we formulate skilled labor as third production factor. We use Two-Level CES function which could provide capital-skill complementarity explicitly. In this paper, BGP exists when the following three conditions hold. First, educational levels are rising over time. Second, capital-augmenting technological change occurs. Finally, $[(c_{KLu}-1)\theta_{Lu}+(c_{KLs}-1)\theta_{Ls}]>0$ and capital-skill complementarity hold where c_{KLu} is the elasticity of complementary between capital and unskilled labor, c_{KLs} is the elasticity of complementary between capital and skilled labor, θ_{Lu} is unskilled labor share, θ_{Ls} is skilled labor share. Grossman et al. (2017a) is a special case in which this paper is formulated by a two factors production function, and the third condition of this paper includes the condition of Grossman et al. (2017a). Moreover, in our model, by defining a three-factor production function, capital-skill complementarity can be clearly represented.

Also, if growth rates of unskilled labor and skilled labor are same, skill-biased technological change is required in addition to the above second and third conditions. Skill-biased technological change is pointed out by Acemoglu (2002) and Card and DiNardo (2002). According to Grossman et al. (2017a), when the education level was constant, there was requires either σ_{KL} or an absence of capital-augmenting technological progress. But, we can represent capital-skill complementarity, capital-augmenting technological progress, and skill-biased technological change.

Grossman et al. (2017a) focus on labor demand and supply in the above analysis. In their model, the inequality between labor was ambiguous. On the other hand, we focus on production technology (labor demand). Therefore, it clearly can represent capital-skill complementarity and skill biased technological change, which are pointed out as cause of the inequality between labor in recent years.

The structure of this article is as follows. In next section, We extend Grossman et al. (2017a) to a three-factors production function of capital, skilled labor and unskilled labor, and we show that exists even when the elasticity of substitution between factors is not unity and capital-biased technological change occurs. In Section 3, we specify the production function as Two-Level Constant Elasticity of Substitution (CES) function. This Two-Level CES function can represent capital-skill complementarity. Section 4 presents the situations in which skill-biased technological change is required for BGP to exist. Finally, the conclusion is stated.

2. Three-Factors Production Function

In this section, We extend Grossman et al. (2017a) to a three-factor production function of capital, skilled labor and unskilled labor. The production function is given as

$$Y_t = F(A_t K_t, B_{ut} L_{ut}, B_{st} L_{st}) \tag{1}$$

where K_t is capital, L_{ut} is unskilled labor, L_{st} is skilled labor, and where A_t, B_{ut}, and B_{st} are the state of technology at time t. Also, L_{ut} and L_{st} grow at constant rate, g_{Lu} and g_{Ls}. We assume that the production function is constant returns to scale in three arguments[6]. Grossman et al. (2017a) discusses two production factors of capital and labor and educational level and this expression is ambiguous. The increase in education level in Grossman et al. (2017a) merely represents an increase in productivity, and the reward for education is unclear. In addition, the expression of capital-skill complementarity is ambiguous because it was discussed under the setting of single labor. In this paper, we have a more rigorous discussion by three factor productions function.

At time t, the economy can convert one unit of output into q_t units of capital. Growth in q_t represents what Greenwood, Hercowitz and Krusell (1997) have called investment-specific technological change[7]. The economy's resource constraint can be written as

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[6] Grossman et al. (2017a) denoted $Y_t = F(A_t K_t, B_t L_t, s_t)$ where K_t is capital, L_t is raw labor, s_t is scalar measure of the prevailing education level in the economy. A_t and B_t are state of technology. They interpreted s_t as the average years of schooling among workers, or the fraction of the labor force with a college degree, or the ratio of trained managers to production-line workers. Although, Grossman et al. (2017b) and Grossman et al. (2020) have endogenized human capital investment, they have similar formulations. In their model, the production function is constant to returns to scale in first two arguments.

[7] According to Greenwood, Hercowitz and Krusell (1997), there are two interpretations of investment-specific technological change. First, q is the productivity of newly installed equipment, and an increase in q indicates that newly installed equipment is more productive than previous ones. For example, a new PC. Second, $1/q$ is the cost of installing new equipment, and an increase in q represents a decrease in

$$Y_t = C_t + I_t/q_t \tag{2}$$

where C_t is consumption and I_t is the number of newly installed units of capital. The capital accumulation equation is

$$\dot{K}_t = I_t - \delta K_t \tag{3}$$

where δ is depletion rate of capital. We define a balanced growth path (BGP) as a trajectory along which experiences constant proportional rates of growth of Y_t, C_t, and K_t. Let $g_X = \dot{X}/X$ denote the growth rate of the variable X along a BGP.

Lemma 1 : Suppose g_q is constant. Then, along any BGP with $0 < C_t < Y_t$, $g_Y = g_C = g_K - g_q$.

See Appendix B for proof of Lemma 1. Lemma 1 is the same as that derived in Grossman et al. (2017a) [8]. Lemma 1 states that the growth rates of consumption and capital mirror that of total output.

We define $\gamma_K \equiv g_A + g_q$ where g_A is disembodied progress, g_q is embodied progress. Thus, γ_K is total rate of capital-augmenting technological change.

Proposition 1 : Suppose q grows at constant rate g_q. If there exists a BGP along which factor shares are constant and strictly positive when the factors are paid their marginal products, then

$$[(c_{KLu} - 1)\theta_{Lu} + (c_{KLs} - 1)\theta_{Ls}]\gamma_K = \theta_{Lu}\theta_{Ls}[c_{KLs} - c_{KLu}]\frac{\dot{s}_t}{s_t}. \tag{4}$$

See Appendix C for proof of Proposition 1, where $c_{KLu} = \frac{F_{12}F}{F_1 F_2}$ is the elasticity of complementary between capital and unskilled labor, $c_{KLs} = \frac{F_{13}F}{F_1 F_3}$ is the elasticity of complementary between capital and skilled labor, θ_{Lu} is unskilled labor share, θ_{Ls} is skilled labor share, F_1, F_2 and F_3 are marginal products of effective capital, effective

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the cost of capital investment. See Appendix A for proof that the final goods price p is equal to the equipment cost $1/q$.

[8] Lemma 1 was proved by Jones and Scrimgeour (2008).

unskilled labor, and effective skilled labor, $s_t \equiv \frac{B_{st}L_{st}}{B_{ut}L_{ut}}$ is ratio of effective skilled labor to effective unskilled labor.

Proposition 1 stipulates a relationship between the total rate of capital-augmenting technological change γ_K and change in ratio of effective skilled labor to effective unskilled labor or education index $\frac{\dot{s}_t}{s_t}$[9] that is needed to keep factor shares constant as the value of the capital stock and output grow at common rate. This is a similar result of Grossman et al. (2017a)[10]. The left side shows the capital-augmenting technological change and the right side shows the change in education level (ratio of skilled labor to unskilled labor) with respect to time. These are the same as Grossman et al. (2017a). On the other hand, by formulating three factors production function, two elasticity of complementary appears on the left side. Grossman et al. (2017a) is a special case in which this paper is formulated by a two factors production function, because elasticity of substitution is inverse of elasticity of complementary with two factor production function. Also, In Grossman et al. (2017a), $\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right)$ on the right side is positive when it is capital-skill complementarity holds. In this paper, capital-skill complementarity holds when $[c_{KLs} - c_{KLu}]$ on the right side is positive.

Both sides of Eq. (4) are positive when the following three conditions hold. First, educational levels are rising over time. This is relevant, as is the increase in the relative supply of skilled labor shown in Acemoglu (2002) and Acemoglu and Autor (2011). Second, capital-augmenting technological change occurs[11]. This is also relevant, as is the decline in investment goods prices shown in Greenwood, Hercowitz and Krusell (1997) and Jones (2016). Finally, $[(c_{KLu} - 1)\theta_{Lu} + (c_{KLs} - 1)\theta_{Ls}] > 0$ and capital-skill complementarity hold.

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9 Acemoglu (2002), Acemoglu and Autor (2011) and McAdam and Willman (2018) defined skilled labor and unskilled labor as college graduate and high school graduate. In this paper, because s_t is ratio of effective skilled labor to effective unskilled labor, it is some as Grossman et al. (2017a).

10 The equivalent of Equation (4) obtained by Grossman et al. (2017a) is $(1 - \sigma_{KL})\gamma_K = \sigma_{KL} \frac{F_L}{F_K} \frac{\partial(F_s/F_L)}{\partial K} \dot{s}$ where F_L is marginal products of raw labor, F_K is marginal products of capital, $\sigma_{KL} \equiv (F_L F_K)/(F_{LK} F)$ is elasticity of substitution between capital and labor, γ_K is total rate of capital-augmenting technological change.

11 In this paper, we have ignored the case where γ_K is negative. When g_A becomes negative (capital using technological progress) and its absolute value is greater thang_q, γ_K becomes negative. In this case, BGP did not theoretically exist in the model of Grossman et al. (2017a), but it can be theoretically shown in this paper.

capital-skill complementarity is empirically supported[12].

3. Two-Level CES Function

In the following, in order to confirm the third condition where both sides of equation (4) are positive, we specify production function (1) as Two-Level Constant Elasticity of Substitution (Two-Level CES) function[13] :

$$Y_t = \left[\mu(B_{ut}L_{ut})^{\frac{\sigma_2-1}{\sigma_2}} + (1-\mu)H^{\frac{\sigma_2-1}{\sigma_2}}\right]^{\frac{\sigma_2}{\sigma_2-1}},$$

$$H_t = \left[\lambda(A_t K_t)^{\frac{\sigma_1-1}{\sigma_1}} + (1-\lambda)(B_{st}L_{st})^{\frac{\sigma_1-1}{\sigma_1}}\right]^{\frac{\sigma_1}{\sigma_1-1}}.$$

H can be interpreted as an intermediate goods or service produced using capital and skilled labor. The final goods Y is produced using unskilled labor and the intermediate goods H. σ_1 is the elasticity of substitution between capital and skilled labor, and σ_2 is the elasticity of substitution between unskilled labor and intermediate goods (, skilled labor, or capital). The elasticity of complementary between factors are

$$c_{KLu} = c_{LuLs} = \frac{1}{\sigma_2},$$

$$c_{KLs} = \frac{1}{1-\theta_{Lu}}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) + \frac{1}{\sigma_2}.$$

Thus, equation (4) gives

$$\left[\frac{1-\sigma_1}{\sigma_1}\theta_{Ls} + \frac{1-\sigma_2}{\sigma_2}\theta_{Lu}\theta_K\right]\gamma_K = \theta_{Lu}\theta_{Ls}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)\frac{\dot{s}_t}{s_t}. \qquad (5)$$

Equation (5) is a special case where equation (4) is specified as a Two-Level CES production function. In two factors production function, $\theta_K = 1 - \theta_{Lu}$, so Grossman et al.

[12] See Goldin and Katz (1998), Krusell et al. (2000) and McAdam and Willman (2018).

[13] See Sato (1967) for Two-Level CES function. The Two-Level CES in this paper is $Y_t = [H(K, L_s), L_u]$, but $Y_t = [H(L_u, L_s), K]$ is also possible. The former provide capital-skill complementarity explicitly. See Goldin and Katz (1998), Krusell et al. (2000), Acemoglu (2002), Papageorgiou and Saam (2008), McAdam and Willman (2018), and Egger and Nigai (2018) for Two-Level CES function which provide capital-skill complementarity.

(2017a) is also a special case of equation (5). When $\left(\frac{1}{\sigma_1}-\frac{1}{\sigma_2}\right)$ on the right-hand side is positive, $\frac{\partial}{\partial K}\left(\frac{F_S}{F_L}\right)$ [14] is positive, so equation (5) provides capital-skill complementarity more explicitly Grossman et al. (2017a). Figure 1 shows $\frac{1}{\sigma_2}$ on the vertical axis and $\frac{1}{\sigma_1}$ on the horizontal axis. Equation

$$\frac{1-\sigma_1}{\sigma_1}\theta_{Ls}+\frac{1-\sigma_2}{\sigma_2}\theta_{Lu}\theta_K=0 \tag{6}$$

is illustrated as this figure (labeled A). Since the area that represents capital-skill complementarity ($\sigma_2>\sigma_1$) is the lower part of the straight line $\sigma_1=\sigma_2$, and the area where the left side of equation (5) is positive is the upper part of the straight line A, so the area where both sides of equation (5) are positive is the area that is drawn dotted line in the figure. Grossman et al. (2017a) did not distinguish between σ_1 and σ_2, so only the upper right part of Figure 1 could be represented, but in the model of this paper, $\sigma_1<1<\sigma_2$ supported in the empirical study of capital-skill complementarity.

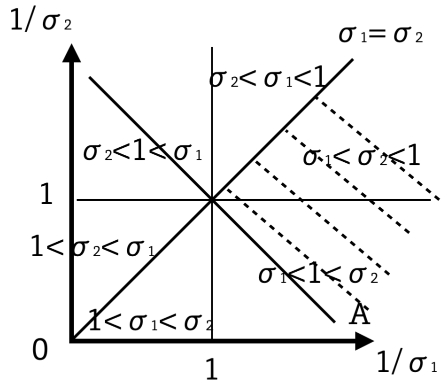

Figure 1：Capital-skill complementarity

Setting $\dot{s}=0$ in equation (5), we obtain Corollary 1.

Corollary 1 : Suppose that s is constant. Then a BGP with constant and strictly positive factor shares can exist only If $\gamma_K=0$ or $\frac{1-\sigma_1}{\sigma_1}\theta_{Ls}+\frac{1-\sigma_2}{\sigma_2}\theta_{Lu}\theta_K=0$.

[14] See Appendix D for the marginal products of unskilled labor and skilled labor with Two-Level CES.

Corollary 1 is also result similar to Grossman et al. (2017a), and balanced growth in a neoclassical economy with exogenous population growth and no investments in human capital requires either an absence of capital-augmenting technological progress or equation (6) holds. In Grossman et al. (2017a), exits of BGP requires either an absence of capital-augmenting technological progress or Cobb-Douglas production function. Cobb-Douglas production function is a case where (6) holds.

Also, Setting $\sigma_1 = \sigma_2 = \sigma$ in equation (5), $\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right) = 0$. This yields Corollary 2.

Corollary 2 : Suppose that $\sigma_1 = \sigma_2 = \sigma$. Then a BGP with constant and strictly positive factor shares can exist only If $\gamma_K = 0$ or $\sigma = 1$.

Corollary 2 is also result similar to Grossman et al. (2017a), but capital-skill complementarity is explicitly shown in this result. In this case, ongoing accumulation of human cannot perpetually neutralize the effects of capital deepening on the factor shares.

4. Skill-Biased Technological Change

In the following, let us consider the case when the growth rates of skilled labor and unskilled labor are the same[15]. In that case, we obtain following proposition.

Proposition 2 : Suppose q grows at constant rate g_q. If there exists a BGP along which factor shares are constant and strictly positive when the factors are paid their marginal products, then

$$\left[\frac{1-\sigma_1}{\sigma_1}\theta_{Ls} + \frac{1-\sigma_2}{\sigma_2}\theta_{Lu}\theta_K\right]\gamma_K = \theta_{Lu}\theta_{Ls}\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)(\gamma_{Ls} - \gamma_{Lu}) \tag{7}$$

[15] With rising levels of education in the economy, it may be unnatural for unskilled and skilled labor to have the same population growth rate. However, the purpose of this section is to theoretically represent the situation in which skill-biased technological change needs to occur in order to BGP exists. Our research focuses on the labor demand side, and it is not an analysis of an economy in which unskilled labor and skilled labor have the same population growth rate.

where $\gamma_{Ls} = \frac{\dot{B}_{st}}{B_{st}}$, and $\gamma_{Lu} = \frac{\dot{B}_{ut}}{B_{ut}}$. When $\gamma_{Ls} > \gamma_{Lu}$ in equation (7), skill-biased technological change occurs. In Grossman et al. (2017a), the right side was zero when the education level was constant.

In order to equation (7) to be positive on both sides, $(\gamma_{Ls} - \gamma_{Lu})$ must be positive in addition to the second and third conditions for both sides of equation (4) to be positive. $(\gamma_{Ls} - \gamma_{Lu}) > 0$ represents skill-biased technological change. It is pointed out by Acemoglu (2002) and Card and DiNardo (2002). In this paper, we can represent capital-skill complementarity, capital-augmenting technological progress, and skill-biased technological change. Also, setting $\gamma_{Ls} = \gamma_{Lu}$ in equation (23) yields Corollary 3.

Corollary 3 : Suppose that growth rates of unskilled labor and skilled labor are same and skill-biased technological change does not occur. Then a BGP with constant and strictly positive factor shares can exist only If $\gamma_K = 0$ or $\sigma_1 = \sigma_2$.

Corollary 3 can not be obtained in Grossman et al. (2017a). In this paper, we can obtain this result by formulating skilled labor as third production factor.

5. Conclusion

In this paper, we showed existence of BGP by extended Grossman et al. (2017a) to three-factor production function of capital, skilled labor and unskilled labor which is specified as a Two-Level CES function. We obtained similar results to their model, even when formulated skill as third production factor. Furthermore, we were able to represent capital-skill complementarity. In this paper, BGP exists when the following three conditions hold. First, educational levels are rising over time. Second, capital-augmenting technological change occurs. Finally, $[(c_{KLu} - 1)\theta_{Lu} + (c_{KLs} - 1)\theta_{Ls}] > 0$ and capital-skill complementarity hold.

Also, if growth rates of unskilled labor and skilled labor are same, skill-biased technological change is required in addition to the above second and third conditions. According to Grossman et al. (2017a), when the education level was constant, there was requires either σ_{KL} or an absence of capital-augmenting technological progress. But, we can represent capital-skill complementarity, capital-augmenting technological progress, and skill-biased technological change.

Appendix A : Proof That the Price of Investment Goods and Final Goods Are Equal

Below we prove that the price of investment goods p_t and the price of final goods $1/q_t$ are equal. Here, it is interpreted that there are firm that convert final goods into investment goods. Thus, suppose there is an investment goods firm that converts $\tilde{I}_t$ to I_t by setting $\tilde{I}_t = I_t/q_t$. The profit of the firm is as follows.

$$\Pi_t = p_t I_t - \tilde{I}_t = p_t q_t \tilde{I}_t - \tilde{I}_t = (p_t q_t - 1)\tilde{I}_t$$

If perfect competition holds, the profit of the firm is zero, so $p_t = 1/q_t$ holds. □

Appendix B : Proof of Lemma 1

In the following, the lemma 1 is proved in the same way as Grossman et al. (2017a). By assumption $C_t < Y_t$, equation (2) ensures $I_t > 0$. Equation (3) implying

$$g_K = \frac{\dot{K}_t}{K_t} = \frac{I_t}{K_t} - \delta. \tag{B1}$$

On a BGP g_K is constant meaning that since $I_t > 0$, $g_I = g_K$.

Differentiating equation (3) with respect to t gives

$$(g_C - g_Y)\frac{C_t}{Y_t} + \left(g_I - g_q - g_Y\right)\frac{I_t/q_t}{Y_t} = 0. \tag{B2}$$

Substituting for $\frac{I_t/q_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ in equation (B2) and using $g_I = g_K$ we have

$$\left(g_K - g_q - g_C\right)\frac{C_t}{Y_t} = \left(g_K - g_q - g_Y\right). \tag{B3}$$

If both sides of this expression equal zero, $g_Y = g_C = g_K - g_q$. Otherwise, since the growth rates are constant on a BGP it must be that C and Y grow at the same rate implying $g_Y = g_C$. But then equation (2) implies $\frac{I_t/q_t}{Y_t} = 1 - \frac{C_t}{Y_t}$ is constant and, since $g_I = g_K$, this ensures $g_Y = g_K - g_q$. □

Appendix C : Proof of Proposition 1

The marginal products of capital, unskilled labor, and skilled labor are

$$F_K=\frac{\partial F(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{\partial K}=\frac{\partial F(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{\partial[AK]}\frac{\partial[AK]}{\partial K}=F_1(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})A_t,$$

$$F_{Lu}=\frac{\partial F(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{\partial L_u}=\frac{\partial F(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{\partial[B_uL_u]}\frac{\partial[B_uL_u]}{\partial L_u}$$
$$=F_2(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})B_{ut},$$

$$F_{Ls}=\frac{\partial F(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{\partial L_s}=\frac{\partial F(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{\partial[B_sL_s]}\frac{\partial[B_sL_s]}{\partial L_s}=F_3(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})B_{st}.$$

Since factors are paid their marginal products the capital, unskilled labor, and skilled labor share are

$$\theta_K=\frac{K_tF_K(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{Y_t},$$

$$\theta_{Lu}=\frac{L_{ut}F_{Lu}(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{Y_t},\theta_{Ls}=\frac{L_{st}F_{Ls}(A_tK_t,B_{ut}L_{ut},B_{st}L_{st})}{Y_t}.$$

From F is constant to returns to scale in three arguments, output per effective unskilled labor is

$$\frac{Y_t}{B_{ut}L_{ut}}=F\left(\frac{A_tK_t}{B_{ut}L_{ut}},\frac{B_{ut}L_{ut}}{B_{ut}L_{ut}},\frac{B_{st}L_{st}}{B_{ut}L_{ut}}\right)=F(k_t,1,s_t)$$

where $k_t\equiv\frac{A_tK_t}{B_{ut}L_{ut}}$ is effective capital per effective unskilled labor, $s_t\equiv\frac{B_{st}L_{st}}{B_{ut}L_{ut}}$ is ratio of effective skilled labor to effective unskilled labor. The marginal products of capital can be written as

$$\frac{\partial[(B_{ut}L_{ut})F(k_t,1,s_t)]}{\partial K}=A_tF_1(k_t,1,s_t).$$

Since on a BGP capital share is constant, differentiating capital share with respect to t gives

$$\frac{\dot{\theta_K}}{\theta_K} = g_A + g_K - g_Y + \frac{d \ln F_1(k_t, 1, s_t)}{dt} = \gamma_K + \frac{d \ln F_1(k_t, 1, s_t)}{dt} = 0. \tag{C1}$$

From F is constant to returns to scale in three arguments, we have

$$aY_t = F(aA_tK_t, aB_{ut}L_{ut}, aB_{st}L_{st}). \tag{C2}$$

Differentiating equation (B2) with respect to a gives

$$Y_t = \frac{\partial F(aA_tK_t, aB_{ut}L_{ut}, aB_{st}L_{st})}{\partial (aA_tK_t)} A_tK_t + \frac{\partial F(aA_tK_t, aB_{ut}L_{ut}, aB_{st}L_{st})}{\partial (aB_{ut}L_{ut})} B_{ut}L_{ut}$$
$$+ \frac{\partial F(aA_tK_t, aB_{ut}L_{ut}, aB_{st}L_{st})}{\partial (aB_{st}L_{st})} B_{st}L_{st}. \tag{C3}$$

Substituting for $a = 1$ in equation (B3), we have

$$Y_t = F_1(A_tK_t, B_{ut}L_{ut}, B_{st}L_{st})A_tK_t + F_2(A_tK_t, B_{ut}L_{ut}, B_{st}L_{st})B_{ut}L_{ut}$$
$$+F_3(A_tK_t, B_{ut}L_{ut}, B_{st}L_{st})B_{st}L_{st} \tag{C4}$$

where F_1, F_2 and F_3 are marginal products of effective capital, effective unskilled labor, and effective skilled labor. The total derivative equation (B4) with respect to Y, K, L_u, and L_s is

$$dY_t = [A_t(F_{11}A_tK_t + F_1) + F_{21}A_tB_{ut}L_{ut} + F_{31}A_tB_{st}L_{st}]dK_t$$
$$+[F_{12}A_tK_tB_{ut} + B_{ut}(F_{22}B_{ut}L_{ut} + F_2) + F_{32}B_{st}L_{st}B_{ut}]dL_{ut}$$
$$+[F_{13}A_tK_tB_{st} + F_{23}B_{ut}L_{ut}B_{st} + B_{st}(F_{33}B_{st}L_{st} + F_3)]dL_{st}. \tag{C5}$$

Also, the total derivative equation (1) with respect to Y, K, L_u, and L_s is

$$dY_t = F_1A_tdK_t + F_2B_{ut}dL_{ut} + F_3B_{st}dL_{st}. \tag{C6}$$

From equation (B5) and (B6), we have

$$(F_{11}A_tK_t + F_{21}B_{ut}L_{ut} + F_{31}B_{st}L_{st})dK_t + (F_{12}A_tK_t + F_{22}B_{ut}L_{ut} + F_{32}B_{st}L_{st})dL_{ut}$$
$$+(F_{13}A_tK_t + F_{23}B_{ut}L_{ut} + F_{33}B_{st}L_{st})dL_{st} = 0. \tag{C7}$$

Using equation (B7), equation (B1) implies

$$\gamma_K = -\frac{d\ln F_1(k_t,1,s_t)}{dt} = -\frac{F_{11}\dot{k}_t + F_{13}\dot{s}_t}{F_1} = \frac{F_{12}}{F_1}\frac{\dot{k}_t}{k_t} + \frac{\left(\frac{\dot{k}_t}{k_t} - \frac{\dot{s}_t}{s_t}\right)s_t F_{13}}{F_1}$$

$$= (c_{KLu}\theta_{Lu} + c_{KLs}\theta_{Ls})\frac{\dot{k}_t}{k_t} - c_{KLs}\theta_{Ls}\frac{\dot{s}_t}{s_t} \quad \text{(C8)}$$

where $c_{KLu} = \frac{F_{12}F}{F_1F_2}$, $c_{LuLs} = \frac{F_{23}F}{F_2F_3}$, and $c_{KLs} = \frac{F_{13}F}{F_1F_3}$ are the elasticity of complementary between capital and unskilled labor, unskilled labor and skilled labor, and capital and skilled labor.

Differentiating equation (C2) with respect to t gives

$$g_Y = \frac{F_1A_tK_t}{F}(g_A + g_K) + \frac{F_2B_{ut}L_{ut}}{F}(g_{Bu} + g_{Lu}) + \frac{F_3B_{st}L_{st}}{F}(g_{Bs} + g_{Ls})$$

$$= \theta_K(g_A + g_K) + \theta_{Lu}(g_{Bu} + g_{Lu}) + \theta_{Ls}(g_{Bs} + g_{Ls})$$

$$= g_A + g_K - (1-\theta_K)\frac{\dot{k}_t}{k_t} + \theta_{Ls}\frac{\dot{s}_t}{s_t}$$

From lemma 1 and definition $\gamma_K \equiv g_A + g_q$, we have

$$\gamma_K = (1-\theta_K)\frac{\dot{k}_t}{k_t} - \theta_{Ls}\frac{\dot{s}_t}{s_t}. \quad \text{(C9)}$$

Finally, substituting equation (B9) for equation (B8), we obtain equation (4). □

Appendix D : The Marginal Products of Unskilled Labor and Skilled Labor

The marginal products of unskilled labor and skilled labor with Two-Level CES are

$$F_{Lu} = \left[\mu(B_{ut}L_{ut})^{\frac{\sigma_2-1}{\sigma_2}} + (1-\mu)H^{\frac{\sigma_2-1}{\sigma_2}}\right]^{\frac{1}{\sigma_2-1}} \mu B_{ut}^{\frac{\sigma_2-1}{\sigma_2}} L_{ut}^{-\frac{1}{\sigma_2}},$$

$$F_{Ls} = \left[\mu(B_{ut}L_{ut})^{\frac{\sigma_2-1}{\sigma_2}} + (1-\mu)H^{\frac{\sigma_2-1}{\sigma_2}}\right]^{\frac{1}{\sigma_2-1}} (1-\mu)$$

$$\times\left[\lambda(A_tK_t)^{\frac{\sigma_1-1}{\sigma_1}}+(1-\lambda)(B_{st}L_{st})^{\frac{\sigma_1-1}{\sigma_1}}\right]^{\frac{\sigma_1(\sigma_2-1)}{\sigma_2(\sigma_1-1)}}(1-\lambda)B_{st}{}^{\frac{\sigma_1-1}{\sigma_1}}L_{st}{}^{-\frac{1}{\sigma_1}}.$$

From above, we have

$$\frac{\partial}{\partial K}\left(\frac{F_s}{F_L}\right)=\frac{(1-\mu)(1-\lambda)B_{st}{}^{\frac{\sigma_1-1}{\sigma_1}}L_{st}{}^{-\frac{1}{\sigma_1}}\lambda A_t{}^{\frac{\sigma_1-1}{\sigma_1}}K_t{}^{-\frac{1}{\sigma_1}}}{\mu B_{ut}{}^{\frac{\sigma_2-1}{\sigma_2}}L_{ut}{}^{-\frac{1}{\sigma_2}}}\frac{\sigma_2-\sigma_1}{\sigma_1\sigma_2}$$
$$\times\left[\lambda(A_tK_t)^{\frac{\sigma_1-1}{\sigma_1}}+(1-\lambda)(B_{st}L_{st})^{\frac{\sigma_1-1}{\sigma_1}}\right]^{\frac{2\sigma_2-\sigma_1\sigma_2-\sigma_1}{\sigma_2(\sigma_1-1)}}.$$

When $\sigma_2 > \sigma_1$ (capital-skill complementarity), this equation is positive. □

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